Lagrangian Formulation Of Underwater Vehicles' Dynamics

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Abstract - Key properties of the equations of motion for underwater vehicles are derived both theoretically and experimentally.
The equations of motion for underwater vehicles are derived in a Lagrangian framework. The Lagrangian approach has several distinctive advantageous to the Newtonian approach. This is especially true in the context of underwater vehicles. The derivation of the hydrodynamic added inertia and the vehicle's rigid body equations of motion can be done in a common framework. The added inertia is given a clear and physical interpretation when we consider the "vehicle-ambient water" system from an energy point of view instead of a force-moment approach.
We have proved that the dynamic equations of motion for an underwater vehicle define a passive mapping between the vehicles thrust $\boldsymbol{\tau}$ and velocity $\dot{\boldsymbol{q}}$.

I. INTRODUCTION

Control system design in robotics is often more simple and intuitive when the dynamic properties of the system is exploited. A typical example is passivity-based control system techniques which exploit the skew-symmetric property of mechanical systems, [6]. This paper shows that this property is also valid for an underwater vehicle where the vehicle's added inertia is also taken into consideration. Lagrangian dynamics is successfully applied to derive these dynamic properties.
This paper is organized as follows. Section II presents the equations of motion for underwater vehicles. Section III describes the Lagrangian formulation for underwater vehicles moving unconstrained in six degrees of freedom. Key properties of the equations of motion are derived in Section IV. Section V gives some concluding remarks.

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II. DYNAMICS OF UNDERWATER VEHICLES

The equations of motion for underwater vehicles can be written as follows [11], [6]:

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{D}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{x}) &= \boldsymbol{w}(\phi) + \boldsymbol{b}(\dot{\boldsymbol{q}}, \boldsymbol{u}) \\ \dot{\boldsymbol{x}} &= \boldsymbol{J}(\boldsymbol{x})\dot{\boldsymbol{q}} \end{aligned} \quad (1)$$

where $\dot{\boldsymbol{q}} = [u, v, w, p, q, r]^T$ is the time derivative of the generalized coordinate vector in the vehicle-fixed reference frame and $\boldsymbol{x} = [x, y, z, \phi, \theta, \psi]^T$ is earth fixed coordinates. Notice that $\boldsymbol{q}$ is a virtual vector. $\boldsymbol{M}$ is the inertia matrix including the hydrodynamic virtual inertia (added mass), $\boldsymbol{C}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ contains the nonlinear forces and moments due to centrifugal and Corolis forces and $\boldsymbol{D}(\dot{\boldsymbol{q}})$ is the vehicles damping matrix, where the potential damping and the viscousus effects are lumped together. $\boldsymbol{g}(\boldsymbol{x})$ is a vector containing the restoring terms formed by the vehicle's buoyancy and gravitational terms. $\boldsymbol{w}(\phi)$ is the wave and current disturbance vector and $\boldsymbol{b}(\dot{\boldsymbol{q}}, \boldsymbol{u})$ is a vector containing the vehicle's propulsion and control forces and moments. $\boldsymbol{J}(\boldsymbol{x})$ is a velocity transformation matrix which transforms velocities from the vehicle-fixed to the earth-fixed reference frame. It should be noted that (1) assumes an irrotational, incompressible and homogeneous fluid of infinite extent.

We propose to use the Lagrangian approach to derive the equations of motion. The Lagrangian approach has several distinctive advantageous to the Newtonian approach used in most textbooks, e.g. [1], [2] and [4]. For instance properties of the inertia matrix and the $\boldsymbol{C}(\dot{\boldsymbol{q}})$ matrix are easier seen in the Lagrangian framework.

III. A LAGRANGIAN APPROACH

A rigid body moving in an unbounded fluid is holonomic by the definition of a rigid body, the unboundness and the infinite extent of the fluid. Hence, a rigid body moving without constraints in six degrees of freedom is an ordinary Lagrangian system. However, the infinite degrees of freedom system formed by the ambient water particles can not be said to be holonomic. A line of reasoning similar to that of [9] will be used to show this. Assume that the solid forms a holonomic system and that all the motion of the ambient water is due to the solid's movements and that it would instantly cease, when the solid is brought to rest. The motion of the water will then be irrotational and acyclic. The ambient water particles should end up in their original position if the solid where moved through a cycle of movements such that it returned to its original position. This is not the case and the ambient water particles do not meet the definition of a holonomic system. However, it is common in the literature, [8] and [3], to also consider the system formed by the surrounding water particles as an Lagrangian system having six degrees of freedom. The proof that the ambient water particles also can be treated as an ordinary Lagrangian system is rather complicated. This proof is made difficult by the fact that the total mass of the water is infinite and that the configuration space of the water particles has infinite dimension, that is the number of degrees of freedom of the ambient water is infinite. A complete proof of this was first presented in [8] and later improved in [3].

We can write the n 2nd order Euler-Lagrange equations for the "rigid body-ambient water" system as

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial F_d}{\partial \dot{q}_i} \quad ; i = 1..n \qquad (2)$$

where Q_i is the generalized forces and $T = T_A + T_{RB}$ is the sum of the ambient water's kinetic energy T_A and the rigid body's kinetic energy T_{RB}. The term $\frac{\partial F_d}{\partial \dot{q}_i}$ represents the dissipative forces on the system. Notice that the potential energy i.e. the effect of gravity, is neglected in the Euler-Lagrange equations. This can be done by applying Avanzini's Theorem which states that the effect of gravity is taken care of by adding a constant hydrostatic buoyancy to the inertial system which will arise without gravity, [3]. This implies that the Lagrangian $L(\dot{\boldsymbol{q}}, \boldsymbol{q}) = T(\dot{\boldsymbol{q}})$ is a so-called inertial Lagrangian system.

A. Kinetic energy of a rigid body

Let the linear and angular velocities be denoted $\boldsymbol{v} = [u, v, w]^T$ and $\boldsymbol{\omega} = [p, q, r]^T$ respectively. Hence, the kinetic energy of a solid in motion, T_{RB}, can be written as

$$\begin{aligned} T_{RB} &= \frac{1}{2}\int\int\int \rho\left(\boldsymbol{v} + (\boldsymbol{\omega}\times\boldsymbol{r})\right)^T\left(\boldsymbol{v} + (\boldsymbol{\omega}\times\boldsymbol{r})\right)\, dV \\ &\Updownarrow \\ T_{RB} &= \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}_{RB}\dot{\boldsymbol{q}} \quad ; \boldsymbol{M}_{RB} = \boldsymbol{M}_{RB}^T > 0\ . \end{aligned}$$

The matrix $\boldsymbol{M}_{RB}$ is found in Appendix A. This can be rewritten to

$$2T_{RB} = \boldsymbol{v}\frac{\partial T_{RB}}{\partial \boldsymbol{v}} + \boldsymbol{\omega}\frac{\partial T_{RB}}{\partial \boldsymbol{\omega}}$$

where the expressions for $\frac{\partial T_{RB}}{\partial \boldsymbol{v}}$ and $\frac{\partial T_{RB}}{\partial \boldsymbol{\omega}}$ are found by comparing the two expressions for T_{RB}.

B. Kinetic energy of the ambient water particles

The kinetic energy of the water particles generated by a moving solid can be written as ,[11],

$$\begin{aligned} T_A &= \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}_A\dot{\boldsymbol{q}} \quad ; \boldsymbol{M}_A = \boldsymbol{M}_A^T > 0 \\ &\Updownarrow \\ T_A &= \boldsymbol{v}\frac{\partial T_A}{\partial \boldsymbol{v}} + \boldsymbol{\omega}\frac{\partial T_A}{\partial \boldsymbol{\omega}}\ . \end{aligned}$$

The $\boldsymbol{M}_A$ matrix is given in Appendix A.

C. Kirchhoff's equations of motion

The equations of motion for the vehicle-fluid system in terms of kinetic energy can be expressed as:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T_{RB}}{\partial \boldsymbol{v}}\right) + \boldsymbol{\omega}\times\frac{\partial T_{RB}}{\partial \boldsymbol{v}} = \\ &\qquad \boldsymbol{\tau}_{1-3} - \frac{d}{dt}\left(\frac{\partial T_A}{\partial \boldsymbol{v}}\right) - \boldsymbol{\omega}\times\frac{\partial T_A}{\partial \boldsymbol{v}} \\ &\frac{d}{dt}\left(\frac{\partial T_{RB}}{\partial \boldsymbol{\omega}}\right) + \boldsymbol{\omega}\times\frac{\partial T_{RB}}{\partial \boldsymbol{\omega}} + \boldsymbol{v}\times\frac{\partial T_{RB}}{\partial \boldsymbol{v}} = \\ &\qquad \boldsymbol{\tau}_{4-6} - \frac{d}{dt}\left(\frac{\partial T_A}{\partial \boldsymbol{\omega}}\right) - \boldsymbol{\omega}\times\frac{\partial T_A}{\partial \boldsymbol{\omega}} - \boldsymbol{v}\times\frac{\partial T_A}{\partial \boldsymbol{v}} \end{aligned}$$

These equations are known as Kirchhoff's equation in vector form, [7]. The left hand side of the equations corresponds to the rigid body's equations of motion. $\boldsymbol{\tau}_{1-6}$ is a vector containing the generalized forces and moments acting on the rigid body, while the rest of the terms on the right hand side of the equations are forces and moments exerted on the rigid body by the fluid pressure.

IV. DERIVATION OF SOME DYNAMIC PROPERTIES

We can without loss of generality simplify (1) to

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{D}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{\tau} \tag{3}$$

where we have assumed still water, no restoring forces present and defined $\boldsymbol{\tau} = \boldsymbol{b}(\dot{\boldsymbol{q}}, \boldsymbol{u})$. This equation has several fundamental properties which may be exploited in regulator design. Recall from Section II that (3) assumes an irrotational, incompressible and homogeneous fluid of infinite extent.

Property 1.

The inertia matrix $\boldsymbol{M}$ is symmetrical and positive definite i.e. $\boldsymbol{M} = \boldsymbol{M}^T > 0$.

Proof:
The kinetic energy of a rigid body with its ambient water particles moving in an ideal homogeneous unbounded fluid of infinite extent is a quadratic function of the vector $\dot{\boldsymbol{q}}$ in the form

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} = \frac{1}{2}\sum_{i,j}^{n} m_{i,j}\dot{q}_i\dot{q}_j$$

where $\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A$. The symmetry property of $\boldsymbol{M}$ is easily seen by interchanging the order of summation such that $T_{ij} = \frac{1}{2}m_{i,j}\dot{q}_i\dot{q}_j = \frac{1}{2}m_{j,i}\dot{q}_i\dot{q}_j$. The positive definiteness is guaranteed by the definition of kinetic energy, that is $T > 0$ if $|\dot{\boldsymbol{q}}| > 0$.

Property 2.

The $\boldsymbol{C}(\dot{\boldsymbol{q}})$ matrix can always be parameterized such that $\boldsymbol{C}(\dot{\boldsymbol{q}})$ becomes skew-symmetrical i.e. $\boldsymbol{C}^T(\dot{\boldsymbol{q}}) + \boldsymbol{C}(\dot{\boldsymbol{q}}) = 0$.

Proof:
Let the $\boldsymbol{C}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ vector be parameterized as

$$\boldsymbol{C}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} \tag{4}$$

Hence it is necessary to prove that $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are skew-symmetrical or nil matrices and sufficient to show that $\boldsymbol{A}_{12}{=}-\boldsymbol{A}_{21}^T$ to prove that $\boldsymbol{C}(\dot{\boldsymbol{q}})$ can be parameterized such that $\boldsymbol{C}(\dot{\boldsymbol{q}})$ becomes skew-symmetrical.
The $\boldsymbol{C}(\dot{\boldsymbol{q}})$ matrix is formed by the cross product terms in Kirchhoff's equation i.e.

$$\mathbf{C}(\dot{q})\dot{q} = \begin{bmatrix} \omega \times \frac{\partial T_{RB}}{\partial v} + \omega \times \frac{\partial T_A}{\partial v} \\ \omega \times \frac{\partial T_{RB}}{\partial \omega} + v \times \frac{\partial T_{RB}}{\partial v} + \omega \times \frac{\partial T_A}{\partial \omega} + v \times \frac{\partial T_A}{\partial v} \end{bmatrix}$$

This expression can be rewritten as

$$\mathbf{C}(\dot{q})\dot{q} = \begin{bmatrix} 0_{3\times 3} & -\left[\left(\frac{\partial T_{RB}}{\partial v} + \frac{\partial T_A}{\partial v}\right)\times\right] \\ -\left[\left(\frac{\partial T_{RB}}{\partial v} + \frac{\partial T_A}{\partial v}\right)\times\right] & -\left[\left(\frac{\partial T_{RB}}{\partial w} + \frac{\partial T_A}{\partial w}\right)\times\right] \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{5}$$

by recognizing the identity $\boldsymbol{a} \times \boldsymbol{Ab} = -\boldsymbol{Ab} \times \boldsymbol{a} = -[\boldsymbol{Ab}\times]\,\boldsymbol{a}$, where the matrix $[\boldsymbol{Ab}\times]$ is skew-symmetrical. We observe that (5) is in the form of (4). It is now trivial to see that $\boldsymbol{A}_{12}{=}-\boldsymbol{A}_{21}^T$, $\boldsymbol{A}_{22}$ is skew-symmetrical and that $\boldsymbol{A}_{11}$ is the nil matrix, which concludes the proof. [3] proves that $\boldsymbol{C}(\dot{\boldsymbol{q}})$ may be parameterized skew-symmetrical from a group theoretical point of view. The $\boldsymbol{C}(\dot{\boldsymbol{q}})$ is found in Appendix A

Remark 1
It is important to notice that the skew-symmetric matrix $\boldsymbol{C}(\dot{\boldsymbol{q}})$ given above is not unique.

Remark 2
Skew-symmetry implies rank deficiency.

Remark 3
Another parameterization than (5) can be found when the rigid body's and the added inertia's kinetic energy is treated separately. For instance

$$\mathbf{C}(\dot{q}) = \begin{bmatrix} [M_{11}\omega\times] & -\left[\left(M_{12}\omega + \frac{\partial T_A}{\partial v}\right)\times\right] \\ -\left[\left(M_{12}\omega + M_{12}v + \frac{\partial T_A}{\partial v}\right)\times\right] & -\left[\left(M_{22}\omega + \frac{\partial T_A}{\partial w}\right)\times\right] \end{bmatrix}$$

where $\boldsymbol{M}_{ij}$ are the 3×3 submatrices of $\boldsymbol{M}_{RB}$ given in Appendix A. The identities $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ and $(\boldsymbol{a}\boldsymbol{c}^T)\boldsymbol{b} = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ can be used to show that $\boldsymbol{A}_{12}{=}-\boldsymbol{A}_{21}^T$.

Property 3.

The property $\dot{\boldsymbol{q}}^T(\dot{\boldsymbol{M}} - 2\boldsymbol{C}(\dot{\boldsymbol{q}}))\dot{\boldsymbol{q}} = 0 \quad \forall \dot{\boldsymbol{q}}$ is always true for any parameterization of $\boldsymbol{C}(\dot{\boldsymbol{q}})$.

Proof:
The proof follows the same line of reasoning as the one employed in [10]. Since $\boldsymbol{M}$ is time-invariant it is necessary and sufficient to prove that $\dot{\boldsymbol{q}}^T\boldsymbol{C}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = 0$ for all possible parameterizations of $\boldsymbol{C}(\dot{\boldsymbol{q}})$ and all possible $\dot{\boldsymbol{q}}$.
The Euler-Lagrange equation $L = (\dot{\boldsymbol{q}}, \boldsymbol{q})$ for a solid moving in an ideal unbounded fluid may be written as an inertial Lagrangian system

$$\boldsymbol{\tau} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}}$$

where the potential energy i.e. the effect of gravity, is neglected by using Avanzini's Theorem, [3]. The Euler-Lagrange equation describe n 2nd order equations. We can transform this to $2n$ first order equations, the so-called Hamilton's equations, by defining the Hamiltonian H and using the Legendre Transformation such that

$$H = \boldsymbol{p}^T \dot{\boldsymbol{q}} - L(\dot{\boldsymbol{q}}).$$

$\boldsymbol{p}$ is the generalized momentum defined as $\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}$. H is on the other hand found to be the sum of kinetic energy and potential energy if we use Lagrange's equation,

$$H = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}}. \tag{6}$$

The $2n$ first order Hamilton's equations are found by employing Lagrange's equations on the Hamiltonian, which yields

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{\partial H}{\partial \boldsymbol{p}} \\ \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{q}} + \boldsymbol{\tau} \ . \end{aligned}$$

The rate of change of the energy in the system defined by the Hamiltonian can now be found as:

$$\frac{dH}{dt} = \frac{\partial H}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} + \frac{\partial H}{\partial \boldsymbol{p}}\dot{\boldsymbol{p}} = \dot{\boldsymbol{q}}^T \boldsymbol{\tau}.$$

Dissipative forces and moments are incorporated by simply subtracting the positive term $\dot{\boldsymbol{q}}^T \boldsymbol{D}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \geq 0$ from the expression, thus

$$\frac{dH}{dt} = \dot{\boldsymbol{q}}^T \left(\boldsymbol{\tau} - \boldsymbol{D}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right). \tag{7}$$

However if we use (6) we obtain

$$\frac{dH}{dt} = \dot{\boldsymbol{q}}^T \boldsymbol{M} \ddot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T \left(\boldsymbol{\tau} - \boldsymbol{C}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{D}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right) \ . \tag{8}$$

We now see that $\dot{\boldsymbol{q}}^T \boldsymbol{C}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ must be zero by comparing (8) and (7).

Remark
Notice that the dissipative term in (7) comes from the term $\frac{\partial F_d}{\partial \dot{q}_i}$ in (2). The dissipative force vector $\boldsymbol{f}_d = \boldsymbol{D}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ can then be derived from $\boldsymbol{D}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = -\frac{\partial F_d}{\partial \dot{\boldsymbol{q}}}$.

Property 4.

The dissipative damping matrix $\boldsymbol{D}(\dot{\boldsymbol{q}})$ is positive semi definite i.e. $\boldsymbol{D}(\dot{\boldsymbol{q}}) \geq 0$.

Proof:
The rate of energy dissipation from the system described in (3) is given by the positive term $\dot{\boldsymbol{q}}^T \boldsymbol{D}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$. $\boldsymbol{D}(\dot{\boldsymbol{q}})$ has to be positive definite to ensure that this term always is positive.

Remark
The matrix $\boldsymbol{D}(\dot{\boldsymbol{q}})$ is positive *semi*-definite only in the case of no dissipative forces.

Definition 1. [13]

A system is said to be passive if it satisfies the following equation:

$$\dot{V}(t) = \boldsymbol{\tau}^T \dot{\boldsymbol{q}} - g(t)$$

where the function $\boldsymbol{V}(t)$ is a Lyapunov function and $g(t) \geq 0$.

Definition 2. [13]

A system is said to be dissipative if it is passive and

$$\int_0^\infty \boldsymbol{\tau}^T \dot{\boldsymbol{q}} \neq 0 \ \Rightarrow \int_0^\infty g(t)dt > 0$$

Property 5.

The dynamic equations for an underwater vehicle define a passive mapping between $\boldsymbol{\tau}$ and $\dot{\boldsymbol{q}}$ written $\boldsymbol{\tau} \rightarrow \dot{\boldsymbol{q}}$.

Proof:
Define a Lyapunov function $V = \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}}$, which is lower bounded. By differentiating V with respect to time we get (assuming $\boldsymbol{M} = \boldsymbol{M}^T$)

$$\dot{V} = \dot{\boldsymbol{q}}^T \boldsymbol{M} \ddot{\boldsymbol{q}} = \boldsymbol{\tau}^T \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T \boldsymbol{D} \dot{\boldsymbol{q}}$$

which prove that the system is passive .

Remark
The system is also dissipative since $\int_0^\infty \dot{\boldsymbol{q}}^T \boldsymbol{D} \dot{\boldsymbol{q}} \ dt > 0$ if $\mid \dot{\boldsymbol{q}} \mid > 0$.

Property 6.

The thrust $\boldsymbol{\tau}$ is a nonlinear function of the control input $\boldsymbol{u}$ and a function of the velocity such that $\boldsymbol{\tau} = \boldsymbol{b}(\boldsymbol{u}, \dot{\boldsymbol{q}})$.

Proof:
Fig. 1 (b) shows some of the test results from an open water test of the Norwegian Experimental Remotely Operated Vehicle's (NEROV) thrusters [12]. The results are presented in the conventional way with the thrust coefficient $K_T = \frac{\tau}{\rho n^2 D^4}$ plotted versus the open water advance coefficient $J_o = \frac{V_A}{nD}$. The advance velocity V_A is defined

as $V_A = (1-w)V$ where w is the wake fraction number (typically 0.05-0.3) and $V = (u^2+v^2+w^2)^{\frac{1}{2}}$ is the vehicle's speed. ρ is density of water, D is propeller diameter and n is the propeller revolution. We have found the K_T values for all four quadrants. This is not common for surface ships, but necessary for a small underwater vehicle which can move in both directions. Notice that the thrust force is almost symmetrical. The thrust τ is related to the K_T coefficient as, [4]:

$$\tau = K_T(J_o)\rho D^4 n \mid n \mid \qquad (9)$$

We observe from Fig. 1 (b) that K_T, in the first and forth quadrant can be approximated by a linear interpolation $K_T = \alpha + \beta J_o$, [5]. Substituting this approximation for K_T into (9) together with the expression for J_o yields

$$\tau_i \approx b_1 u \mid u \mid + b_2\, \dot{q}_i \mid u \mid$$

where $b_1 = \rho D^4 \alpha$ and $b_2 = \rho D^3(1-w)\beta$. The input u is simply n, i.e. the propeller revolution. Hence we have shown that $\boldsymbol{\tau} = \boldsymbol{b}(\boldsymbol{u}, \dot{\boldsymbol{q}})$. This is clearly verified by looking on Fig. 1 (a) where the measured thrust is plotted as a function of different speeds of advance and propeller revolutions.

1 V. CONCLUDING REMARKS

Lagrangian dynamics has successfully been applied to derive some useful properties for underwater vehicles. We have showed that the dynamic equations for an underwater vehicle define a passive mapping between the control input $\boldsymbol{u}$ and the output $\dot{\boldsymbol{q}}$. We have also shown both experimentally and theoretically that the control input $\boldsymbol{\tau}$ for an underwater vehicle with propellers is a nonlinear function of the vehicle's speed $\dot{\boldsymbol{q}}$ and propeller angular revolution n.

These properties will be exploited in future control system design.

Appendix A

The mass matrix $\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A$ is the sum of the rigid body's inertia and the added inertia due to the acceleration of the ambient water particles.

$$\mathbf{M}_{RB} = \begin{bmatrix} m & 0 & 0 & 0 & mz_G & -my_G \\ 0 & m & 0 & -mz_G & 0 & -mx_G \\ 0 & 0 & m & my_G & mx_G & 0 \\ 0 & -mz_G & my_G & I_x & -I_{xy} & -I_{xz} \\ mz_G & 0 & -mx_G & -I_{xy} & I_y & -I_{yz} \\ -my_G & mx_G & 0 & -I_{xz} & -I_{yz} & I_z \end{bmatrix}$$

$$\mathbf{M}_A = -\begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ X_{\dot{v}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ X_{\dot{w}} & Y_{\dot{w}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ X_{\dot{p}} & Y_{\dot{p}} & Z_{\dot{p}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ X_{\dot{q}} & Y_{\dot{q}} & Z_{\dot{q}} & K_{\dot{q}} & M_{\dot{q}} & M_{\dot{r}} \\ X_{\dot{r}} & Y_{\dot{r}} & Z_{\dot{r}} & K_{\dot{r}} & M_{\dot{r}} & N_{\dot{r}} \end{bmatrix}$$

Notice that the added inertia matrix is by convension defined negative definite, ([14]).

The matrix $\boldsymbol{C}(\dot{\boldsymbol{q}}) = \boldsymbol{C}_A(\dot{\boldsymbol{q}}) + \boldsymbol{C}_{RB}(\dot{\boldsymbol{q}})$, [5], [11].

$\mathbf{C}_{RB}(\dot{\boldsymbol{q}}) =$

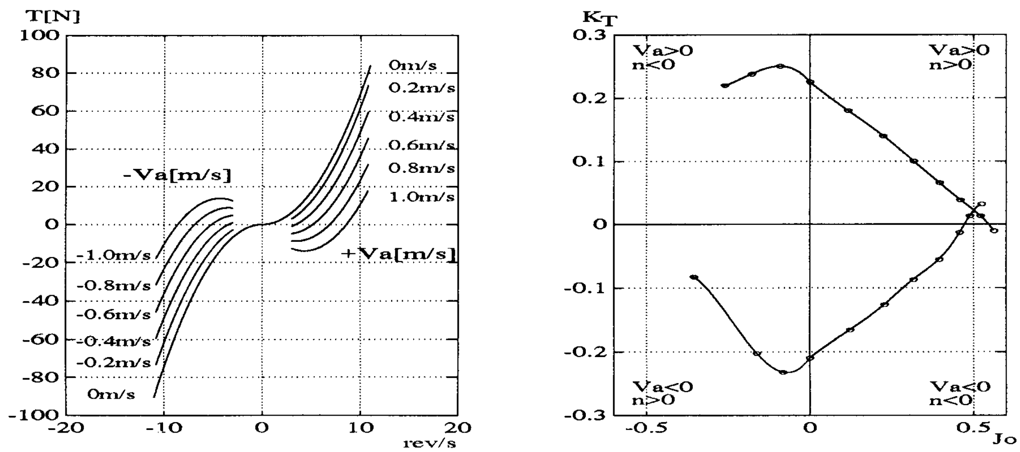
$$\left[\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -m(y_Gq+z_Gr) & m(y_Gp+w) & m(z_Gp-v) \\ m(x_Gq-w) & -m(z_Gr+x_Gp) & m(z_Gq+u) \\ m(x_Gr+v) & m(y_Gr-u) & -m(x_Gp+y_Gq) \end{matrix}\right.$$

$$\left.\begin{matrix} m(y_Gq+z_Gr) & -m(x_Gq-w) & -m(x_Gr+v) \\ -m(y_Gp+w) & m(z_Gr+x_Gp) & -m(y_Gr-u) \\ -(mz_Gp-v) & -m(z_Gq+u) & m(x_Gp+y_Gq) \\ 0 & -I_{yz}q-I_{xz}p+I_zr & I_{yz}r+I_{xy}p-I_yq \\ I_{yz}q+I_{xz}p-I_zr & 0 & -I_{xz}r-I_{xy}q+I_xp \\ -I_{yz}r-I_{xy}p+I_yq & I_{xz}r+I_{xy}q-I_xp & 0 \end{matrix}\right]$$

$$\mathbf{C}_A(\dot{q}) = \begin{bmatrix} 0 & 0 & 0 & 0 & C_A^{15} & C_A^{16} \\ 0 & 0 & 0 & C_A^{24} & 0 & C_A^{26} \\ 0 & 0 & 0 & C_A^{34} & C_A^{35} & 0 \\ 0 & -C_A^{24} & -C_A^{34} & 0 & C_A^{45} & C_A^{46} \\ -C_A^{15} & 0 & -C_A^{35} & -C_A^{45} & 0 & C_A^{56} \\ -C_A^{16} & -C_A^{26} & 0 & -C_A^{46} & -C_A^{56} & 0 \end{bmatrix}$$

where

$$\begin{aligned} C_A^{15} &= -X_{\dot{w}}u - Y_{\dot{w}}v - Z_{\dot{w}}w - Z_{\dot{p}}p - Z_{\dot{q}}q - Z_{\dot{r}}r \\ C_A^{35} &= X_{\dot{u}}u + X_{\dot{v}}v + X_{\dot{w}}w + X_{\dot{p}}p + X_{\dot{q}}q + X_{\dot{r}}r \\ C_A^{16} &= X_{\dot{v}}u + Y_{\dot{v}}v + Y_{\dot{w}}w + Y_{\dot{p}}p + Y_{\dot{q}}q + Y_{\dot{r}}r \\ C_A^{45} &= -X_{\dot{r}}u - Y_{\dot{r}}v - Z_{\dot{r}}w - K_{\dot{r}}p - M_{\dot{r}}q - N_{\dot{r}}r \\ C_A^{24} &= X_{\dot{w}}u + Y_{\dot{w}}v + Z_{\dot{w}}w + Z_{\dot{p}}p + Z_{\dot{q}}q + Z_{\dot{r}}r \\ C_A^{46} &= X_{\dot{q}}u + Y_{\dot{q}}v + Z_{\dot{q}}w + K_{\dot{q}}p + M_{\dot{q}}q + M_{\dot{r}}r \\ C_A^{26} &= -X_{\dot{u}}u - X_{\dot{v}}v - X_{\dot{w}}w - X_{\dot{p}}p - X_{\dot{q}}q - X_{\dot{r}}r \\ C_A^{56} &= -X_{\dot{p}}u - Y_{\dot{p}}v - Z_{\dot{p}}w - K_{\dot{p}}p - K_{\dot{q}}q - K_{\dot{r}}r \\ C_A^{34} &= -X_{\dot{v}}u - Y_{\dot{v}}v - Y_{\dot{w}}w - Y_{\dot{p}}p - Y_{\dot{q}}q - Y_{\dot{r}}r \end{aligned}$$



(a) (b)

Figure 1: *(a) Measured thruster force τ as a function of propeller revolutions n for different speeds of advance V_A.(b) Non-dimensional thrust characteristics K_T versus the advance coefficient J_o for the NEROV.*

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