Nonlinear Control of Dynamic Positioned Ships Using Only Position Feedback: An Observer Backstepping Approach

(Revised Version)

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Abstract

Dynamic positioning (DP) systems for ships are usually designed under the assumption that the kinematic equations can be linearized about a constant yaw angle such that linear theory can be applied. This paper proposes a globally uniformly asymptotically stable (GUAS) nonlinear control law where this assumption is removed. A nonlinear observer is included in the design such that only position measurements are required. GUAS is proven by applying the backstepping design methodology and Lyapunov stability theory. The control law is simulated on a thruster controlled ship.

1 Introduction

Conventional DP-systems for ships are designed by linearizing the kinematic equations of motions about different yaw angles such that linear optimal control theory and a gain-scheduling technique can be applied. In addition to this, a Kalman filter is used to produce noise-free estimates of the velocities and positions when only positions are measured, see [1], [2], [3] and [4].

The main motivation for this paper is to remove these assumptions by using nonlinear observer and feedback control theory. Linearization of the kinematics is avoided by using a modified version of the nonlinear observer presented in [5]. Next, nonlinear feedback from the state estimates is obtained by using the observer backstepping design methodology [6]. The results of [6] are further improved by replacing the measured output with a filtered output when designing the feedback control. Hence, the control inputs are generated by using filtered estimates of both the velocities and positions. Finally, GUAS is proven for the total system (ship model, observer and control system).

2 Ship Model

The nonlinear ship model is based on [7].

2.1 Kinematics

The position $(x,\ y)$ and yaw angle ψ of the vessel is expressed in the earth-fixed reference frame while the surge, sway and yaw velocities (u, v, r) are expressed in the body-fixed frame. Hence:

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{\nu} \tag{1}$$

where $\boldsymbol{\eta} = [\,x\,,\ y\,,\ \psi\,]^T$, $\boldsymbol{\nu} = [\,u\,,\ v\,,r\,]^T$ and:

$$\boldsymbol{J}(\boldsymbol{\eta}) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

2.2 Dynamics

The case study of this paper is an anchored ship equipped with thrusters for DP. The mooring forces are modeled as spring forces, $\boldsymbol{K}(\boldsymbol{\eta} - \boldsymbol{\eta}_0)$ where $\boldsymbol{\eta}_0$ is the equilibrium position of the vessel. For simplicity it is assumed that $\boldsymbol{\eta}_0 = \mathbf{0}$. Thus, the body-fixed equations of motion in surge, sway and yaw can be written:

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{D}\boldsymbol{\nu} + \boldsymbol{K}\boldsymbol{\eta} = \boldsymbol{\tau} \tag{3}$$

Here, $\boldsymbol{\tau} = [\,\tau_1\,,\ \tau_2\,,\ \tau_3\,]^T$ is a control vector of forces and moment in surge, sway and yaw provided by the thruster system. The matrices $\mathbf{M}, \mathbf{D}$ and $\mathbf{K}$ for a typical floating production ship are given in Appendix B.

2.3 Resulting System Model

The resulting system model is written as:

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\psi)\boldsymbol{\nu} && (4) \\ \dot{\boldsymbol{\nu}} &= -\boldsymbol{A}_1\boldsymbol{\eta} - \boldsymbol{A}_2\boldsymbol{\nu} + \boldsymbol{B}\boldsymbol{\tau} && (5) \end{aligned}$$

where

$$\boldsymbol{A}_1 = \boldsymbol{M}^{-1}\boldsymbol{K}, \quad \boldsymbol{A}_2 = \boldsymbol{M}^{-1}\boldsymbol{D}, \quad \boldsymbol{B} = \boldsymbol{M}^{-1} \tag{6}$$

It is assumed that only position and heading measurements $\boldsymbol{\eta}$ are available.

3 Nonlinear Observer Design

The nonlinear observer is found by using Lyapunov theory which puts constraints on the choice of the filter gains. This is based on [5], where a nonlinear model-based observer for an underwater vehicle with filter gains which are functions of the measured attitude are proposed. The yaw angle ψ is assumed to be measured with good accuracy by using a gyro compass. The position measurements x and y are assumed measured by using DGPS. An observer for (4)–(5) is constructed as:

$$\dot{\hat{\boldsymbol{\eta}}} = \boldsymbol{J}(\psi)\hat{\boldsymbol{\nu}} + \boldsymbol{K}_1\tilde{\boldsymbol{\eta}} \quad (7)$$

$$\dot{\hat{\boldsymbol{\nu}}} = -\boldsymbol{A}_1\hat{\boldsymbol{\eta}} - \boldsymbol{A}_2\hat{\boldsymbol{\nu}} + \boldsymbol{B}\boldsymbol{\tau} + \boldsymbol{K}_2\tilde{\boldsymbol{\eta}} \quad (8)$$

where $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta} - \hat{\boldsymbol{\eta}}$ is the position estimation error. If, $\tilde{\boldsymbol{\nu}} = \boldsymbol{\nu} - \hat{\boldsymbol{\nu}}$, the total error system becomes:

$$\dot{\tilde{\boldsymbol{\eta}}} = \boldsymbol{J}(\psi)\tilde{\boldsymbol{\nu}} - \boldsymbol{K}_1\tilde{\boldsymbol{\eta}} \quad (9)$$

$$\dot{\tilde{\boldsymbol{\nu}}} = -\boldsymbol{A}_1\tilde{\boldsymbol{\eta}} - \boldsymbol{A}_2\tilde{\boldsymbol{\nu}} - \boldsymbol{K}_2\tilde{\boldsymbol{\eta}} \quad (10)$$

Notice that the measured yaw angle ψ is used to compute $\boldsymbol{J}(\psi)$ while $\hat{\psi}$ is used for state feedback. The computation of $\boldsymbol{J}(\psi)$ is quite accurate since the gyro compass measurement noise will be less than 0.1 (deg) most of the time. However, good filtering of x and y is important since DGPS-measurement noise will be in the range of 1–3 (m).

The matrices $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ in (9)–(10) must be chosen such that the observer is GUAS. This is obtained by defining a Lyapunov function candidate:

$$V_{\text{obs}}(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\nu}}) = \frac{1}{2}\left(\tilde{\boldsymbol{\eta}}^T\boldsymbol{P}_1\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\nu}}^T\boldsymbol{P}_2\tilde{\boldsymbol{\nu}}\right) \quad (11)$$

where $\boldsymbol{P}_1 = \boldsymbol{P}_1^T$ and $\boldsymbol{P}_2 = \boldsymbol{P}_2^T$ are positive definite matrices. Hence:

$$\begin{aligned} \dot{V}_{\text{obs}} &= \dot{\tilde{\boldsymbol{\eta}}}^T\boldsymbol{P}_1\tilde{\boldsymbol{\eta}} + \frac{1}{2}(\tilde{\boldsymbol{\nu}}^T\boldsymbol{P}_2\dot{\tilde{\boldsymbol{\nu}}} + \dot{\tilde{\boldsymbol{\nu}}}^T\boldsymbol{P}_2\tilde{\boldsymbol{\nu}}) \\ &= (\boldsymbol{J}(\psi)\tilde{\boldsymbol{\nu}} - \boldsymbol{K}_1\tilde{\boldsymbol{\eta}})^T\boldsymbol{P}_1\tilde{\boldsymbol{\eta}} \\ &\quad + \frac{1}{2}\tilde{\boldsymbol{\nu}}^T\boldsymbol{P}_2(-\boldsymbol{A}_1\tilde{\boldsymbol{\eta}} - \boldsymbol{A}_2\tilde{\boldsymbol{\nu}} - \boldsymbol{K}_2\tilde{\boldsymbol{\eta}}) \quad (12) \\ &\quad + \frac{1}{2}(-\boldsymbol{A}_1\tilde{\boldsymbol{\eta}} - \boldsymbol{A}_2\tilde{\boldsymbol{\nu}} - \boldsymbol{K}_2\tilde{\boldsymbol{\eta}})^T\boldsymbol{P}_2\tilde{\boldsymbol{\nu}} \\ &= \tilde{\boldsymbol{\nu}}^T(\boldsymbol{J}^T(\psi)\boldsymbol{P}_1 - \boldsymbol{P}_2\boldsymbol{A}_1 - \boldsymbol{P}_2\boldsymbol{K}_2)\tilde{\boldsymbol{\eta}} \\ &\quad - \tilde{\boldsymbol{\eta}}^T\boldsymbol{K}_1^T\boldsymbol{P}_1\tilde{\boldsymbol{\eta}} - \frac{1}{2}\tilde{\boldsymbol{\nu}}^T(\boldsymbol{P}_2\boldsymbol{A}_2 + \boldsymbol{A}_2^T\boldsymbol{P}_2)\tilde{\boldsymbol{\nu}} \end{aligned}$$

$\dot{V}_{\text{obs}}$ can be made negative definite by defining:

$$\boldsymbol{J}^T(\psi)\boldsymbol{P}_1 - \boldsymbol{P}_2\boldsymbol{A}_1 - \boldsymbol{P}_2\boldsymbol{K}_2 \triangleq \boldsymbol{0} \quad (13)$$

$$\boldsymbol{K}_1^T\boldsymbol{P}_1 \triangleq \boldsymbol{Q}_1 \quad (14)$$

$$\frac{1}{2}(\boldsymbol{P}_2\boldsymbol{A}_2 + \boldsymbol{A}_2^T\boldsymbol{P}_2) \triangleq \boldsymbol{Q}_2 \quad (15)$$

where $\boldsymbol{Q}_1 = \boldsymbol{Q}_1^T$ and $\boldsymbol{Q}_2 = \boldsymbol{Q}_2^T$ are positive definite design matrices. Hence:

$$\dot{V}_{\text{obs}} = -\tilde{\boldsymbol{\eta}}^T\boldsymbol{Q}_1\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\nu}}^T\boldsymbol{Q}_2\tilde{\boldsymbol{\nu}} < 0, \quad \forall\tilde{\boldsymbol{\eta}} \neq \boldsymbol{0}, \tilde{\boldsymbol{\nu}} \neq \boldsymbol{0} \quad (16)$$

which proves that the observer is GUAS. The definitions (13)–(15) are satisfied if:

$$\boldsymbol{K}_1 = \boldsymbol{P}_1^{-1}\boldsymbol{Q}_1 \quad (17)$$

$$\boldsymbol{K}_2(\psi) = \boldsymbol{P}_2^{-1}\boldsymbol{J}^T(\psi)\boldsymbol{P}_1 - \boldsymbol{A}_1 \quad (18)$$

Notice that $\mathbf{K}_2$ is an explicit function of ψ. Also notice that only $\boldsymbol{P}_1$, $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are design matrices since $\boldsymbol{P}_2$ is given by (15).

4 Velocity and Position Observer Backstepping

In this section a GUAS nonlinear control law using the observer in Section 3 is derived. The observer (7)–(8) can be written in component form:

$$\begin{aligned} \dot{\hat{x}}_1 &= \cos x_3 \cdot \hat{x}_4 - \sin x_3 \cdot \hat{x}_5 + k_1\tilde{x}_1 + k_2\tilde{x}_2 + k_3\tilde{x}_3 \\ \dot{\hat{x}}_2 &= \sin x_3 \cdot \hat{x}_4 + \cos x_3 \cdot \hat{x}_5 + k_4\tilde{x}_1 + k_5\tilde{x}_2 + k_6\tilde{x}_3 \\ \dot{\hat{x}}_3 &= \hat{x}_6 + k_7\tilde{x}_1 + k_8\tilde{x}_2 + k_9\tilde{x}_3 \\ \dot{\hat{x}}_4 &= -a_1\hat{x}_1 - a_2\hat{x}_4 + b_1u_1 + k_{10}\tilde{x}_1 + k_{11}\tilde{x}_2 + k_{12}\tilde{x}_3 \\ \dot{\hat{x}}_5 &= -a_3\hat{x}_2 - a_4\hat{x}_5 - a_5\hat{x}_6 \\ &\quad + b_2u_2 + b_3u_3 + k_{13}\tilde{x}_1 + k_{14}\tilde{x}_2 + k_{15}\tilde{x}_3 \\ \dot{\hat{x}}_6 &= -a_6\hat{x}_2 - a_7\hat{x}_5 - a_8\hat{x}_6 \\ &\quad + b_4u_2 + b_5u_3 + k_{16}\tilde{x}_1 + k_{17}\tilde{x}_2 + k_{18}\tilde{x}_3 \quad (19) \end{aligned}$$

where the matrix elements are defined according to:

$$\boldsymbol{K}_1 = \begin{bmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{bmatrix} \quad \boldsymbol{K}_2 = \begin{bmatrix} k_{10} & k_{11} & k_{12} \\ k_{13} & k_{14} & k_{15} \\ k_{16} & k_{17} & k_{18} \end{bmatrix}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & a_6 & 0 \end{bmatrix} \quad \boldsymbol{A}_2 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_4 & a_5 \\ 0 & a_7 & a_8 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & b_3 \\ 0 & b_4 & b_5 \end{bmatrix}$$

The position and attitude variables x_1, x_2 and x_3 (x, y and ψ) are the variables in interest of controlling. Let $c_i > 0$ and $d_i > 0$ ($i = 1 \ldots 6$) be 12 design constants to be defined later.

Step 1: The first tracking objective is $x_1 - x_{1d}$. Since x_1, x_2 and x_3 are measured with sensor noise, they are replaced by their estimates and the error variable is defined as $z_1 = \hat{x}_1 - x_{1d}$ where x_{1d} is the desired state. Thus, using (19), $\dot{z}_1$ can be written as

$$\begin{aligned} \dot{z}_1 &= \cos x_3 \cdot \hat{x}_4 - \sin x_3 \cdot \hat{x}_5 \quad (20) \\ &\quad + k_1\tilde{x}_1 + k_2\tilde{x}_2 + k_3\tilde{x}_3 - \dot{x}_{1d} \end{aligned}$$

The idea of backstepping is to choose one of the state variables as virtual controls. Equation (20) does not imply an obvious choice of which variable to choose as virtual control. However, we suggest:

$$\xi_1 = \cos x_3 \cdot \hat{x}_4 - \sin x_3 \cdot \hat{x}_5 \quad (21)$$

The estimation errors $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ in (20) are treated as unknown disturbances which requires that the stabilizing function α_1 must include an additional damping term. Moreover:

$$\alpha_1 = -c_1z_1 - d_1(k_1^2 + k_2^2 + k_3^2)z_1 + \dot{x}_{1d} \quad (22)$$

This specific choice will prove to be very useful when stability is investigated later on in this section.

Step 2: The error variable of the first virtual control is defined as:

$$\begin{aligned} z_2 &= \xi_1 - \alpha_1 \quad (23) \\ &= \cos x_3 \cdot \hat{x}_4 - \sin x_3 \cdot \hat{x}_5 - \alpha_1 \end{aligned}$$

Substitution of (23) into (20), yields:

$$\begin{aligned} \dot{z}_1 &= -c_1 z_1 + k_1\tilde{x}_1 + k_2\tilde{x}_2 + k_3\tilde{x}_3 \\ &\quad - d_1(k_1^2 + k_2^2 + k_3^2)z_1 + z_2 \end{aligned} \tag{24}$$

Time differentiation and reordering of (23) under the assumption that k_1, k_2 and k_3 are constants, see (17), yields:

$$\begin{aligned} \dot{z}_2 &= \dot{\xi}_1 - \dot{\alpha}_1 \\ &= \cos x_3 b_1 u_1 - \sin x_3 b_2 u_2 - \sin x_3 b_3 u_3 \\ &\quad - \sin x_3(\hat{x}_4\hat{x}_6 - a_3\hat{x}_2 - a_4\hat{x}_5 - a_5\hat{x}_6) \\ &\quad + \cos x_3(-\hat{x}_5\hat{x}_6 - a_1\hat{x}_1 - a_2\hat{x}_4) - \ddot{x}_{1d} \\ &\quad - c_1^2 z_1 - 2c_1 d_1(k_1^2+k_2^2+k_3^2)z_1 + c_1 z_2 \\ &\quad - d_1^2(k_1^2+k_2^2+k_3^2)^2 z_1 + d_1(k_1^2+k_2^2+k_3^2)z_2 \\ &\quad + (k_{10}\cos x_3 - k_{13}\sin x_3 \\ &\qquad + k_1 c_1 + k_1 d_1(k_1^2+k_2^2+k_3^2))\tilde{x}_1 \\ &\quad + (k_{11}\cos x_3 - k_{14}\sin x_3 \\ &\qquad + k_2 c_1 + k_2 d_1(k_1^2+k_2^2+k_3^2))\tilde{x}_2 \\ &\quad + (k_{12}\cos x_3 - k_{15}\sin x_3 \\ &\qquad + k_3 c_1 + k_3 d_1(k_1^2+k_2^2+k_3^2))\tilde{x}_3 \\ &\quad - (\sin x_3 \cdot \hat{x}_4 + \cos x_3 \cdot \hat{x}_5)\tilde{x}_6 \\ &\triangleq \alpha_{21} b_1 u_1 + \alpha_{22} b_2 u_2 + \alpha_{23} b_3 u_3 + \psi_2 \\ &\quad + \omega_{21}\tilde{x}_1 + \omega_{22}\tilde{x}_2 + \omega_{23}\tilde{x}_3 + \omega_{26}\tilde{x}_6 \end{aligned} \tag{25}$$

where α_{21}, α_{22}, α_{23}, ω_{24}, ω_{25} and ω_{26} are given in Appendix A. The feedback control is chosen without using the error terms $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$, that is:

$$\begin{aligned} &\alpha_{21} b_1 u_1 + \alpha_{22} b_2 u_2 + \alpha_{23} b_3 u_3 \\ &\qquad = -c_2 z_2 - z_1 - \psi_2 \\ &\qquad\quad - d_2(\omega_{21}^2 + \omega_{22}^2 + \omega_{23}^2 + \omega_{26}^2)z_2 \end{aligned} \tag{26}$$

which substituted into (25) gives:

$$\begin{aligned} \dot{z}_2 &= -c_2 z_2 + \omega_{21}\tilde{x}_1 + \omega_{22}\tilde{x}_2 + \omega_{23}\tilde{x}_3 + \omega_{26}\tilde{x}_6 \\ &\quad - d_2(\omega_{21}^2 + \omega_{22}^2 + \omega_{23}^2 + \omega_{26}^2)z_2 - z_1 \end{aligned} \tag{27}$$

The feedback control in (26) is only one equation to be satisfied by the three controls u_1, u_2 and u_3. The next steps will find two more equations for the control variables to satisfy.

Step 3: The next tracking objective is $z_3 = \hat{x}_2 - x_{2d}$ where x_{2d} is the desired path of x_2:

$$\begin{aligned} \dot{z}_3 &= \sin x_3 \cdot \hat{x}_4 + \cos x_3 \cdot \hat{x}_5 \\ &\quad + k_4\tilde{x}_1 + k_5\tilde{x}_2 + k_6\tilde{x}_3 - \dot{x}_{2d} \end{aligned} \tag{28}$$

This is very similar to (20), and the choices of virtual control ξ_3 and the stabilizing function α_3 can be done in the same way:

$$\xi_3 = \sin x_3 \cdot \hat{x}_4 + \cos x_3 \cdot \hat{x}_5 \tag{29}$$

$$\alpha_3 = -c_3 z_3 - d_3(k_4^2 + k_5^2 + k_6^2)z_3 + \dot{x}_{2d} \tag{30}$$

Step 4: The virtual control has the error variable

$$z_4 = \xi_3 - \alpha_3 \tag{31}$$

which substituted into (28) using (29) and (30), yields (assuming k_4, k_5 and k_6 constant, see (17)):

$$\begin{aligned} \dot{z}_3 &= -c_3 z_3 + k_4\tilde{x}_1 + k_5\tilde{x}_2 + k_6\tilde{x}_3 \\ &\quad - d_3(k_4^2 + k_5^2 + k_6^2)z_3 + z_4 \end{aligned} \tag{32}$$

Hence, $\dot{z}_4$ is found by time differentiation and reordering of (31), that is:

$$\begin{aligned} \dot{z}_4 &= \dot{\xi}_3 - \dot{\alpha}_3 \\ &= \sin x_3 b_1 u_1 + \cos x_3 b_2 u_2 + \cos x_3 b_3 u_3 \\ &\quad + \sin x_3(-\hat{x}_5\hat{x}_6 - a_1\hat{x}_1 - a_2\hat{x}_4) \\ &\quad + \cos x_3(\hat{x}_4\hat{x}_6 - a_3\hat{x}_2 - a_4\hat{x}_5 - a_5\hat{x}_6) - \ddot{x}_{2d} \\ &\quad - c_3^2 z_3 - 2c_3 d_3(k_4^2+k_5^2+k_6^2)z_3 + c_3 z_4 \\ &\quad - d_3^2(k_4^2+k_5^2+k_6^2)^2 z_3 + d_3(k_4^2+k_5^2+k_6^2)z_4 \\ &\quad + (k_{10}\sin x_3 + k_{13}\cos x_3 + k_4 c_3 \\ &\qquad + k_4 d_3(k_4^2+k_5^2+k_6^2))\tilde{x}_1 \\ &\quad + (k_{11}\sin x_3 + k_{14}\cos x_3 + k_5 c_3 \\ &\qquad + k_5 d_3(k_4^2+k_5^2+k_6^2))\tilde{x}_2 \\ &\quad + (k_{12}\sin x_3 + k_{15}\cos x_3 + k_6 c_3 \\ &\qquad + k_6 d_3(k_4^2+k_5^2+k_6^2))\tilde{x}_3 \\ &\quad + (\cos x_3 \cdot \hat{x}_4 - \sin x_3 \cdot \hat{x}_5)\tilde{x}_6 \\ &\triangleq \alpha_{41} b_1 u_1 + \alpha_{42} b_2 u_2 + \alpha_{43} b_3 u_3 + \psi_4 \\ &\quad + \omega_{41}\tilde{x}_1 + \omega_{42}\tilde{x}_5 + \omega_{43}\tilde{x}_3 + \omega_{46}\tilde{x}_6 \end{aligned} \tag{33}$$

See Appendix A for the definitions of the terms in (33). Next, the following choice of feedback control is made:

$$\begin{aligned} &\alpha_{41} b_1 u_1 + \alpha_{42} b_2 u_2 + \alpha_{43} b_3 u_3 \\ &\qquad = -c_4 z_4 - z_3 - \psi_4 \\ &\qquad\quad - d_4(\omega_{41}^2 + \omega_{42}^2 + \omega_{43}^2 + \omega_{46}^2)z_4 \end{aligned} \tag{34}$$

which is the second equation to be satisfied by the three controls u_1, u_2 and u_3. Finally, the equation for $\dot{z}_4$ can be derived by substituting (34) into (33):

$$\begin{aligned} \dot{z}_4 &= -c_4 z_4 + \omega_{41}\tilde{x}_1 + \omega_{42}\tilde{x}_2 + \omega_{43}\tilde{x}_3 + \omega_{46}\tilde{x}_6 \\ &\quad - d_4(\omega_{41}^2 + \omega_{42}^2 + \omega_{43}^2 + \omega_{46}^2)z_4 - z_3 \end{aligned} \tag{35}$$

Step 5: $\hat{x}_3$ obtained from the observer can be used for state feedback. The tracking error is $z_5 = \hat{x}_3 - x_{3d}$:

$$\dot{z}_5 = \hat{x}_6 - \dot{x}_{3d} \tag{36}$$

Hence, we can choose:

$$\xi_5 = \hat{x}_6 \tag{37}$$

$$\alpha_5 = -c_5 z_5 + \dot{x}_{3d} - d_5(k_7^2 + k_8^2 + k_9^2)z_5 \tag{38}$$

Step 6: The virtual control has the error variable $z_6 = \xi_5 - \alpha_5$ which leads to:

$$\begin{aligned} \dot{z}_5 &= -c_5 z_5 + k_7\tilde{x}_1 + k_8\tilde{x}_2 + k_9\tilde{x}_3 \\ &\quad - d_5(k_7^2 + k_8^2 + k_9^2)z_5 + z_6 \end{aligned} \tag{39}$$

$$
\begin{aligned}
\dot{z}_6 &= \dot{\xi}_5 - \dot{\alpha}_5 \\
&= b_4 u_2 + b_5 u_3 \qquad (40) \\
&\quad - a_6 \hat{x}_2 - a_7 \hat{x}_5 - a_8 \hat{x}_6 \\
&\quad - c_5^2 z_5 - 2 c_5 d_5 (k_7^2 + k_8^2 + k_9^2) z_5 + c_5 z_6 \\
&\quad - (d_5^2 z_5 + d_5 z_6)(k_7^2 + k_8^2 + k_9^2) \\
&\quad + (k_{16} + k_7 c_5 + k_7 d_5 (k_7^2 + k_8^2 + k_9^2)) \tilde{x}_1 \\
&\quad + (k_{17} + k_8 c_5 + k_8 d_5 (k_7^2 + k_8^2 + k_9^2)) \tilde{x}_2 \\
&\quad + (k_{18} + k_9 c_5 + k_9 d_5 (k_7^2 + k_8^2 + k_9^2)) \tilde{x}_3 \\
&\triangleq +\alpha_{62} b_4 u_2 + \alpha_{63} b_5 u_3 + \psi_6 \\
&\quad + \omega_{61} \tilde{x}_1 + \omega_{62} \tilde{x}_2 + \omega_{63} \tilde{x}_3
\end{aligned}
$$

The following choice of feedback is made:

$$
\begin{aligned}
&\alpha_{62} b_2 u_4 + \alpha_{63} b_5 u_3 \qquad (41) \\
&= -c_6 z_6 - z_5 - \psi_6 - d_6 (\omega_{61}^2 + \omega_{62}^2 + \omega_{63}^2) z_6
\end{aligned}
$$

and the final equation for $\dot{z}_6$ can be found:

$$
\begin{aligned}
\dot{z}_6 &= -c_6 z_6 + \omega_{61} \tilde{x}_1 + \omega_{62} \tilde{x}_2 + \omega_{63} \tilde{x}_3 \\
&\quad - d_6 (\omega_{61}^2 + \omega_{62}^2 + \omega_{63}^2) z_6 - z_5 \qquad (42)
\end{aligned}
$$

Resulting Control Law: The previous outline found three equations that the control variables must satisfy. Solving (26), (34) and (41) for u_1, u_2 and u_3, the following control law is obtained:

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -\boldsymbol{S}^{-1} \begin{bmatrix} c_2 z_2 + \psi_2 + d_2 (\omega_{21}^2 + \omega_{22}^2 + \omega_{23}^2 + \omega_{26}^2) z_2 + z_1 \\ c_4 z_4 + \psi_4 + d_4 (\omega_{41}^2 + \omega_{42}^2 + \omega_{43}^2 + \omega_{46}^2) z_4 + z_3 \\ c_6 z_6 + \psi_6 + d_6 (\omega_{61}^2 + \omega_{62}^2 + \omega_{63}^2) z_6 + z_5 \end{bmatrix} \qquad (43)
$$

where

$$
\boldsymbol{S} = \begin{bmatrix} \alpha_{21} b_1 & \alpha_{22} b_2 & \alpha_{23} b_3 \\ \alpha_{41} b_1 & \alpha_{42} b_2 & \alpha_{43} b_3 \\ 0 & \alpha_{62} b_4 & \alpha_{63} b_5 \end{bmatrix} \qquad (44)
$$

Feasibility and Stability: The control law (43) is well defined only if the matrix $\boldsymbol{S}$ is invertible for all $\boldsymbol{x}$. This can be further investigated by computing:

$$
\begin{aligned}
\det(\boldsymbol{S}) &= \begin{vmatrix} \cos x_3 \cdot b_1 & -\sin x_3 \cdot b_2 & -\sin x_3 \cdot b_3 \\ \sin x_3 \cdot b_1 & \cos x_3 \cdot b_2 & \cos x_3 \cdot b_3 \\ 0 & b_4 & b_5 \end{vmatrix} \\
&= b_1 (b_2 b_5 - b_3 b_4) = \det(\boldsymbol{M}^{-1}) > 0 \qquad (45)
\end{aligned}
$$

Moreover $\det(\boldsymbol{S}) > 0$ since inertia is a positive quantity. Hence, the matrix $\boldsymbol{S}$ is non-singular for all x_3. Stability of the observer and controller can be proven by considering the resulting state-space model:

$$
\begin{aligned}
\dot{z}_1 &= -c_1 z_1 + k_1 \tilde{x}_1 + k_2 \tilde{x}_2 + k_3 \tilde{x}_3 \\
&\quad - d_1 (k_1^2 + k_2^2 + k_3^2) z_1 + z_2 \\
\dot{z}_2 &= -c_2 z_2 + \omega_{21} \tilde{x}_1 + \omega_{22} \tilde{x}_2 + \omega_{23} \tilde{x}_3 + \omega_{26} \tilde{x}_6 \\
&\quad - d_2 (\omega_{21}^2 + \omega_{22}^2 + \omega_{23}^2 + \omega_{26}^2) z_2 - z_1 \\
\dot{z}_3 &= -c_3 z_3 + k_4 \tilde{x}_1 + k_5 \tilde{x}_2 + k_6 \tilde{x}_3 \\
&\quad - d_3 (k_4^2 + k_5^2 + k_6^2) z_3 + z_4 \\
\dot{z}_4 &= -c_4 z_4 + \omega_{41} \tilde{x}_1 + \omega_{42} \tilde{x}_2 + \omega_{43} \tilde{x}_3 + \omega_{46} \tilde{x}_6 \\
&\quad - d_4 (\omega_{41}^2 + \omega_{42}^2 + \omega_{43}^2 + \omega_{46}^2) z_4 - z_3 \\
\dot{z}_5 &= -c_5 z_5 + k_7 \tilde{x}_1 + k_8 \tilde{x}_2 + k_9 \tilde{x}_3 \\
&\quad - d_5 (k_7^2 + k_8^2 + k_9^2) z_5 + z_6 \\
\dot{z}_6 &= -c_6 z_6 + \omega_{61} \tilde{x}_1 + \omega_{62} \tilde{x}_2 + \omega_{63} \tilde{x}_3 \qquad (46) \\
&\quad - d_6 (\omega_{61}^2 + \omega_{62}^2 + \omega_{63}^2) z_6 - z_5 \\
\dot{\tilde{x}}_1 &= \cos x_3 \cdot \tilde{x}_4 - \sin x_3 \cdot \tilde{x}_5 - k_1 \tilde{x}_1 - k_2 \tilde{x}_2 - k_3 \tilde{x}_3 \\
\dot{\tilde{x}}_2 &= \sin x_3 \cdot \tilde{x}_4 + \cos x_3 \cdot \tilde{x}_5 - k_4 \tilde{x}_1 - k_5 \tilde{x}_2 - k_6 \tilde{x}_3 \\
\dot{\tilde{x}}_3 &= \hat{x}_6 - k_7 \tilde{x}_1 - k_8 \tilde{x}_2 - k_9 \tilde{x}_3 \\
\dot{\tilde{x}}_4 &= -a_1 \tilde{x}_1 - a_2 \tilde{x}_4 - k_{10} \tilde{x}_1 - k_{11} \tilde{x}_2 - k_{12} \tilde{x}_3 \\
\dot{\tilde{x}}_5 &= -a_3 \tilde{x}_2 - a_4 \tilde{x}_5 - a_5 \tilde{x}_6 - k_{13} \tilde{x}_1 - k_{14} \tilde{x}_2 - k_{15} \tilde{x}_3 \\
\dot{\tilde{x}}_6 &= -a_6 \tilde{x}_3 - a_7 \tilde{x}_5 - a_8 \tilde{x}_6 - k_{16} \tilde{x}_1 - k_{17} \tilde{x}_2 - k_{18} \tilde{x}_3
\end{aligned}
$$

which can be rewritten in vector form as:

$$
\begin{aligned}
\dot{\boldsymbol{z}} &= -\boldsymbol{C}_z \boldsymbol{z} - \boldsymbol{D}_z \boldsymbol{z} + \boldsymbol{E}_z \boldsymbol{z} + \boldsymbol{W}_{\tilde{\eta}} \tilde{\boldsymbol{\eta}} + \boldsymbol{W}_{\tilde{\nu}} \tilde{\boldsymbol{\nu}} \qquad (47) \\
\dot{\tilde{\boldsymbol{\eta}}} &= \boldsymbol{J}(x_3) \tilde{\boldsymbol{\nu}} - \boldsymbol{K}_1 \tilde{\boldsymbol{\eta}} \qquad (48) \\
\dot{\tilde{\boldsymbol{\nu}}} &= -\boldsymbol{A}_1 \tilde{\boldsymbol{\eta}} - \boldsymbol{A}_2 \tilde{\boldsymbol{\nu}} - \boldsymbol{K}_2 \tilde{\boldsymbol{\eta}} \qquad (49)
\end{aligned}
$$

where $\boldsymbol{z} = [z_1, ..., z_6]^T$, $\tilde{\boldsymbol{\eta}} = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]^T$, $\tilde{\boldsymbol{\nu}} = [\tilde{x}_4, \tilde{x}_5, \tilde{x}_6]^T$ and

$$
\begin{aligned}
\boldsymbol{C}_z &= \operatorname{diag}(c_1, c_2, c_3, c_4, c_5, c_6) \qquad (50) \\
\boldsymbol{D}_z &= \operatorname{diag}\big(d_1 (k_1^2 + k_2^2 + k_3^2), \\
&\quad d_2 (\omega_{21}^2 + \omega_{22}^2 + \omega_{23}^2 + \omega_{26}^2), d_3 (k_4^2 + k_5^2 + k_6^2), \\
&\quad d_4 (\omega_{41}^2 + \omega_{42}^2 + \omega_{43}^2 + \omega_{46}^2), d_5 (k_7^2 + k_8^2 + k_9^2), \\
&\quad d_6 (\omega_{61}^2 + \omega_{62}^2 + \omega_{63}^2)\big) \qquad (51)
\end{aligned}
$$

$$
\boldsymbol{E}_z = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad (52)
$$

$$
\boldsymbol{W}_{\tilde{\eta}} = \begin{bmatrix} k_1 & k_2 & k_3 \\ \omega_{21} & \omega_{22} & \omega_{23} \\ k_4 & k_5 & k_6 \\ \omega_{41} & \omega_{42} & \omega_{43} \\ k_7 & k_8 & k_9 \\ \omega_{61} & \omega_{62} & \omega_{63} \end{bmatrix} \boldsymbol{W}_{\tilde{\nu}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_{26} \\ 0 & 0 & 0 \\ 0 & 0 & \omega_{46} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (53)
$$

A Lyapunov function candidate is defined as:

$$
V_{\text{con}} = \frac{1}{2} \left(\boldsymbol{z}^T \boldsymbol{z} + \tilde{\boldsymbol{\eta}}^T \boldsymbol{P}_1 \tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\nu}}^T \boldsymbol{P}_2 \tilde{\boldsymbol{\nu}} \right) \qquad (54)
$$

where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are positive definite matrices. Time differentiation of (54) gives:

$$
\begin{aligned}
\dot{V}_{\text{con}} &= \boldsymbol{z}^T \dot{\boldsymbol{z}} + \dot{\tilde{\boldsymbol{\eta}}}^T \boldsymbol{P}_1 \tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\nu}}^T \boldsymbol{P}_2 \dot{\tilde{\boldsymbol{\nu}}} \\
&= \boldsymbol{z}^T (-\boldsymbol{C}_z \boldsymbol{z} - \boldsymbol{D}_z \boldsymbol{z} + \boldsymbol{E}_z \boldsymbol{z} + \boldsymbol{W}_{\tilde{\eta}} \tilde{\boldsymbol{\eta}} + \boldsymbol{W}_{\tilde{\nu}} \tilde{\boldsymbol{\nu}}) \\
&\quad + (\boldsymbol{J}(x_3) \tilde{\boldsymbol{\nu}} - \boldsymbol{K}_1 \tilde{\boldsymbol{\eta}})^T \boldsymbol{P}_1 \tilde{\boldsymbol{\eta}} \\
&\quad + \tilde{\boldsymbol{\nu}}^T \boldsymbol{P}_2 (-\boldsymbol{A}_1 \tilde{\boldsymbol{\eta}} - \boldsymbol{A}_2 \tilde{\boldsymbol{\nu}} - \boldsymbol{K}_2 \tilde{\boldsymbol{\eta}}) \\
&= -\boldsymbol{z}^T \boldsymbol{C}_z \boldsymbol{z} + \boldsymbol{z}^T \boldsymbol{E}_z \boldsymbol{z} \\
&\quad - \boldsymbol{z}^T \boldsymbol{D}_z \boldsymbol{z} + \boldsymbol{z}^T \boldsymbol{W}_{\tilde{\eta}} \tilde{\boldsymbol{\eta}} + \boldsymbol{z}^T \boldsymbol{W}_{\tilde{\nu}} \tilde{\boldsymbol{\nu}} \\
&\quad - \frac{1}{4} \tilde{\boldsymbol{\eta}}^T \boldsymbol{G}_1 \tilde{\boldsymbol{\eta}} - \frac{1}{4} \tilde{\boldsymbol{\nu}}^T \boldsymbol{G}_2 \tilde{\boldsymbol{\nu}} - \tilde{\boldsymbol{\eta}}^T (\boldsymbol{K}_1^T \boldsymbol{P}_1 - \frac{1}{4} \boldsymbol{G}_1) \tilde{\boldsymbol{\eta}} \\
&\quad - \tilde{\boldsymbol{\nu}}^T (\boldsymbol{P}_2 \boldsymbol{A}_2 - \frac{1}{4} \boldsymbol{G}_2) \tilde{\boldsymbol{\nu}} \\
&\quad - \tilde{\boldsymbol{\nu}}^T (\boldsymbol{P}_2 \boldsymbol{A}_1 + \boldsymbol{P}_2 \boldsymbol{K}_2 - \boldsymbol{J}^T(x_3) \boldsymbol{P}_1) \tilde{\boldsymbol{\eta}} \qquad (55)
\end{aligned}
$$

where we have added the zero terms $\frac{1}{4}(\tilde{\boldsymbol{\eta}}^T \boldsymbol{G}_1 \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}}^T \boldsymbol{G}_1 \tilde{\boldsymbol{\eta}})$ and $\frac{1}{4}(\tilde{\boldsymbol{\nu}}^T \boldsymbol{G}_2 \tilde{\boldsymbol{\nu}} - \tilde{\boldsymbol{\nu}}^T \boldsymbol{G}_2 \tilde{\boldsymbol{\nu}})$, with:

$$\boldsymbol{G}_1 = \mathrm{diag}(\, g_1 \,, g_1 \,, g_1 \,) \tag{56}$$

$$\boldsymbol{G}_2 = \mathrm{diag}(\, 0 \,, 0 \,, g_2 \,) \tag{57}$$

$$g_1 = \sum_{i=1}^{6} \frac{1}{d_i} > 0 \tag{58}$$

$$g_2 = \frac{1}{d_2} + \frac{1}{d_4} > 0 \tag{59}$$

The matrix $\boldsymbol{E}_z$ in (52) is skew-symmetrical. Hence, $\boldsymbol{z}^T \boldsymbol{E}_z \boldsymbol{z} = 0 \; \forall \; \boldsymbol{z}$. It is also seen that:

$$\begin{array}{l} -\boldsymbol{z}^T \boldsymbol{D}_z \boldsymbol{z} + \boldsymbol{z}^T \boldsymbol{W}_{\tilde{\eta}} \tilde{\boldsymbol{\eta}} + \boldsymbol{z}^T \boldsymbol{W}_{\tilde{\nu}} \tilde{\boldsymbol{\nu}} \\ \qquad -\frac{1}{4}(\tilde{\boldsymbol{\eta}}^T \boldsymbol{G}_1 \tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\nu}}^T \boldsymbol{G}_2 \tilde{\boldsymbol{\nu}}) \leq 0 \end{array} \tag{60}$$

or equivalently:

$$\begin{array}{l}
-d_1(\frac{1}{2d_1}\tilde{x}_1 - z_1 k_1)^2 - d_1(\frac{1}{2d_1}\tilde{x}_2 - z_1 k_2)^2 \\
-d_1(\frac{1}{2d_1}\tilde{x}_3 - z_1 k_3)^2 - d_2(\frac{1}{2d_2}\tilde{x}_1 - z_2 \omega_{21})^2 \\
-d_2(\frac{1}{2d_2}\tilde{x}_2 - z_2 \omega_{22})^2 - d_2(\frac{1}{2d_2}\tilde{x}_3 - z_2 \omega_{23})^2 \\
-d_2(\frac{1}{2d_2}\tilde{x}_6 - z_2 \omega_{26})^2 - d_3(\frac{1}{2d_3}\tilde{x}_1 - z_3 k_4)^2 \\
-d_3(\frac{1}{2d_3}\tilde{x}_2 - z_3 k_5)^2 - d_3(\frac{1}{2d_3}\tilde{x}_3 - z_3 k_6)^2 \\
-d_4(\frac{1}{2d_4}\tilde{x}_1 - z_4 \omega_{41})^2 - d_4(\frac{1}{2d_4}\tilde{x}_2 - z_4 \omega_{42})^2 \\
-d_4(\frac{1}{2d_4}\tilde{x}_3 - z_4 \omega_{43})^2 - d_4(\frac{1}{2d_4}\tilde{x}_6 - z_4 \omega_{46})^2 \\
-d_5(\frac{1}{2d_5}\tilde{x}_1 - z_5 k_7)^2 - d_5(\frac{1}{2d_5}\tilde{x}_2 - z_5 k_8)^2 \\
-d_5(\frac{1}{2d_5}\tilde{x}_3 - z_5 k_9)^2 - d_6(\frac{1}{2d_6}\tilde{x}_1 - z_6 \omega_{61})^2 \\
-d_6(\frac{1}{2d_6}\tilde{x}_2 - z_6 \omega_{62})^2 - d_6(\frac{1}{2d_6}\tilde{x}_3 - z_6 \omega_{63})^2 \leq 0
\end{array} \tag{61}$$

In addition, the observer choice (18) for $\boldsymbol{K}_2$ makes the last term in (55) equal to zero. Substituting the observer gain (17) into (55) yields:

$$\begin{array}{rcl} \dot{V}_{\mathrm{con}} & \leq & -\boldsymbol{z}^T \boldsymbol{C}_z \boldsymbol{z} - \tilde{\boldsymbol{\eta}}^T (\boldsymbol{Q}_1 - \frac{1}{4}\boldsymbol{G}_1) \tilde{\boldsymbol{\eta}} \\ & & - \tilde{\boldsymbol{\nu}}^T (\boldsymbol{Q}_2 - \frac{1}{4}\boldsymbol{G}_2) \tilde{\boldsymbol{\nu}} \end{array} \tag{62}$$

Finally, $\boldsymbol{Q}_1 - \frac{1}{4}\boldsymbol{G}_1$ and $\boldsymbol{Q}_2 - \frac{1}{4}\boldsymbol{G}_2$ must be proven to be positive definite. Since $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ defined in (56) and (57) are diagonal matrices, this can easily be obtained by choosing $||\boldsymbol{Q}_1|| > \frac{1}{4}||\boldsymbol{G}_1||$ and $||\boldsymbol{Q}_2|| > \frac{1}{4}||\boldsymbol{G}_2||$. The matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ also have to be chosen such that $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are positive definite matrices. Hence, according to the *LaSalle-Yoshizawa* theorem [6] the system (47)–(49) with (43) is GUAS. Notice that $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are not needed for implementation.

5 Simulation Studies

The control law (43) with observer (7) and (8) was simulated with $\boldsymbol{C}_z = \boldsymbol{I}$ and $d_i = 0.01$ (i=1...6). The observer gains $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ were computed by using (17) and (18) with $\boldsymbol{P}_1 = \mathrm{diag}(400, 300, 300)$, $\boldsymbol{Q}_1 = \mathrm{diag}(1000, 1500, 4000)$ and $\boldsymbol{Q}_2 = \mathrm{diag}(100, 200, 300) \cdot \boldsymbol{A}_2$. The sampling time was 0.1 s. The simulation results are shown in Figures 1 and 2.

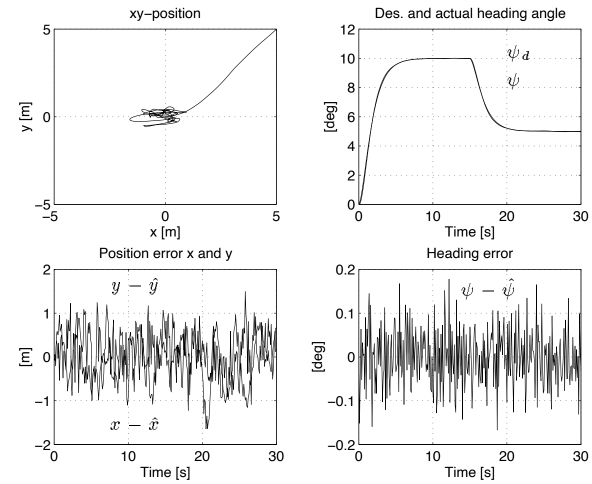

Figure 1: Upper plots: Simultaneously regulation of positions (x,y) to (0,0) and tracking of yaw angle ψ to ψ_d. Lower plots: Observer estimation errors.

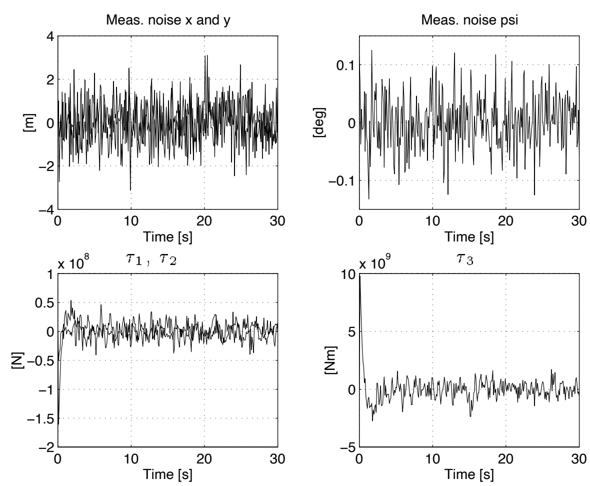

Figure 2: Upper plots: Measurement noise for x, y and ψ. Lower plots: Control inputs τ_1, τ_2 and τ_3.

6 Conclusions

A globally uniformly asymptotically stable (GUAS) nonlinear control law with observer for dynamic positioning (DP) of ships have been derived. The nonlinear control law and observer are based on the nonlinear kinematic equations of motion to describe the position and yaw angle of the vessel. Hence, linearization and gain-scheduling techniques can be avoided when designing the control law. A nonlinear observer is used to produce noise-free estimates of velocity and posi-

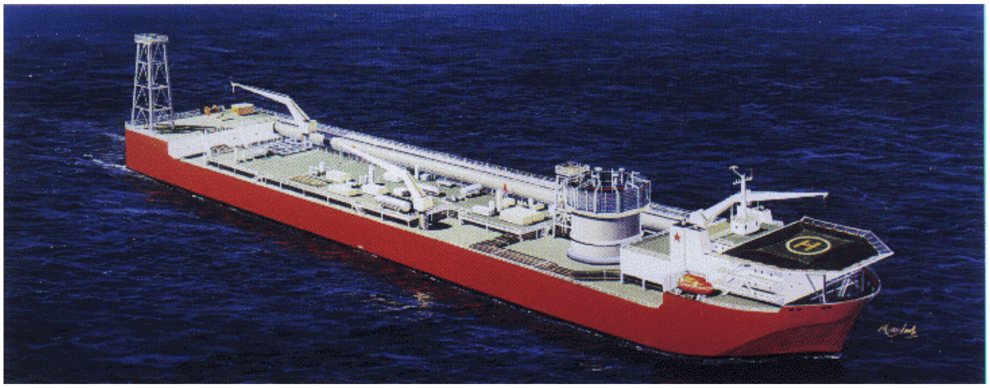

Figure 3: Moored ship: Length $L = 200.6$ (m) and mass $m = 73.097.150$ (kg).

tion from noisy position measurements. The sensors considered are a gyro compass and a GPS-receiver. GUAS was proven by applying the backstepping design methodology and Lyapunov stability theory.

A Tables

Tables 1–3 lists the functions used in (43).

Table 1: α_{ij}-functions.

$\alpha_{21} = \cos x_3$	$\alpha_{41} = \sin x_3$	
$\alpha_{22} = -\sin x_3$	$\alpha_{42} = \cos x_3$	$\alpha_{62} = 1$
$\alpha_{23} = -\sin x_3$	$\alpha_{43} = \cos x_3$	$\alpha_{63} = 1$

Table 2: ψ_i-functions.

$\psi_2 = -\sin x_3(\hat{x}_4\hat{x}_6 - a_3\hat{x}_2 - a_4\hat{x}_5 - a_5\hat{x}_6) + \cos x_3(-\hat{x}_5\hat{x}_6 - a_1\hat{x}_1 - a_2\hat{x}_4) - \ddot{x}_{1d} - c_1^2 z_1 - 2c_1 d_1(k_1^2 + k_2^2 + k_3^2)z_1 + c_1 z_2 - d_1^2(k_1^2 + k_2^2 + k_3^2)^2 z_1 + d_1(k_1^2 + k_2^2 + k_3^2)z_2$
$\psi_4 = \sin x_3(-\hat{x}_5\hat{x}_6 - a_1\hat{x}_1 - a_2\hat{x}_4) + \cos x_3(\hat{x}_4\hat{x}_6 - a_3\hat{x}_2 - a_4\hat{x}_5 - a_5\hat{x}_6) - \ddot{x}_{2d} - c_3^2 z_3 - 2c_3 d_3(k_4^2 + k_5^2 + k_6^2)z_3 + c_3 z_4 - d_3^2(k_4^2 + k_5^2 + k_6^2)^2 z_3 + d_3(k_4^2 + k_5^2 + k_6^2)z_4$
$\psi_6 = -a_6\hat{x}_2 - a_7\hat{x}_5 - a_8\hat{x}_6 - c_5^2 z_5 - 2c_5 d_5(k_7^2 + k_8^2 + k_9^2)z_5 + c_5 z_6 - d_5^2(k_7^2 + k_8^2 + k_9^2)^2 z_5 + d_5(k_7^2 + k_8^2 + k_9^2)z_6$

Table 3: ω_{ij}-functions.

$\omega_{21} = k_{10}\cos x_3 - k_{13}\sin x_3 + k_1 c_1 + k_1 d_1(k_1^2 + k_2^2 + k_3^2)$
$\omega_{22} = k_{11}\cos x_3 - k_{14}\sin x_3 + k_2 c_1 + k_2 d_1(k_1^2 + k_2^2 + k_3^2)$
$\omega_{23} = k_{12}\cos x_3 - k_{15}\sin x_3 + k_3 c_1 + k_3 d_1(k_1^2 + k_2^2 + k_3^2)$
$\omega_{26} = -\sin x_3 \cdot \hat{x}_4 - \cos x_3 \cdot \hat{x}_5$
$\omega_{41} = k_{10}\sin x_3 + k_{13}\cos x_3 + k_4 c_3 + k_4 d_3(k_4^2 + k_5^2 + k_6^2)$
$\omega_{42} = k_{11}\sin x_3 + k_{14}\cos x_3 + k_5 c_3 + k_5 d_3(k_4^2 + k_5^2 + k_6^2)$
$\omega_{43} = k_{12}\sin x_3 + k_{15}\cos x_3 + k_6 c_3 + k_6 d_3(k_4^2 + k_5^2 + k_6^2)$
$\omega_{46} = \cos x_3 \cdot \hat{x}_4 - \sin x_3 \cdot \hat{x}_5$
$\omega_{61} = k_{16} + k_7 c_5 + k_7 d_5(k_7^2 + k_8^2 + k_9^2)$
$\omega_{62} = k_{17} + k_8 c_5 + k_7 d_5(k_7^2 + k_8^2 + k_9^2)$
$\omega_{63} = k_{18} + k_9 c_5 + k_7 d_5(k_7^2 + k_8^2 + k_9^2)$

B Data for Moored Ship

The system matrices for the moored ship shown in Figure 3 are scaled according to the *Bis-system* [7]:

$$\boldsymbol{M} = \begin{bmatrix} 1.0852 & 0 & 0 \\ 0 & 2.0575 & -0.4087 \\ 0 & -0.4087 & 0.2153 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} 0.0865 & 0 & 0 \\ 0 & 0.0762 & 0.1510 \\ 0 & 0.0151 & 0.0031 \end{bmatrix}$$

$$\boldsymbol{K} = \mathrm{diag}(0.0389, 0.0266, 0)$$

For DP it can be shown that the system matrices satisfies [7]; $\boldsymbol{M} = \boldsymbol{M}^T > \boldsymbol{0}$, $\dot{\boldsymbol{M}} = \boldsymbol{0}$ and $\boldsymbol{\nu}^T \boldsymbol{D}\boldsymbol{\nu} > 0$.

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