SEMI-GLOBAL RESULTS ON STABILIZATION OF LINEAR SYSTEMS WITH INPUT RATE AND MAGNITUDE SATURATIONS

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Abstract

This paper gives some new semi-global results for linear systems with simultaneous magnitude and rate saturations. In particular, linear control laws giving a locally attractive equilibrium are derived. It is also shown that for null-controllable systems the domain of attraction can be made arbitrary large. For unstable systems, necessary conditions for the existence of a linear controller resulting in a local attractive equilibrium are given. Moreover, for a large class of unstable systems the domain of attraction can be made arbitrary large in some directions of the state-space. Two examples illustrating the results are also included.

1 Introduction

Some of the most common and significant nonlinearities in control systems are those of the actuator and then, in particular, magnitude and rate saturations. These nonlinearities are present in virtually all physical systems, even though the degree of importance may vary. In some control systems the input nonlinearities may be negligible, while in other their effects may be significant, leading to stability problems or aggravation of performance.

There has been a renewed interest in the problem of input saturation the last few years, and some interesting results have been reported. These results address primarily the problem of global and semi-global stabilization using bounded control. The notion of semi-global exponential stability of linear systems subject to input saturation was first introduced by Lin and Saberi[4], and has later been further developed, see e.g. Saberi *et al.*[6] and Teel[10]. The global stabilization problem for an integrator chain is impossible to solve using linear control laws. However, by using a nonlinear control law of nested saturation functions, global stability was proven by Teel[9]. This result was later generalized by Sussman[8] to null-controllable plants.

All of the mentioned results addresses the pure magnitude saturation problem, and not the problem of input rate saturation. It is clear that rate saturation is more difficult to handle than a pure magnitude saturation. This is due to the fact that while limits in the magnitude only changes the gain in the closed-loop system, rate limits have memory, and thus affects both the magnitude and the phase. One approach to solve this problem is to augment the system with one integrator and then derive a stabilizing control law for the time derivative of the input. This method usually requires input measurements and the augmented system will have state constraints. Most design methods are unable to handle this with the results of Sussman[8] and Teel[11] as important exceptions.

When dealing with input nonlinearities like magnitude and/or rate saturation, the problem is often evaded by generating less aggressive reference trajectories and hence avoiding saturating actuators, see e.g. Miller and Pachter[5]. From the practical point of view, the existing results are mostly based on simulations or *ad hoc* solutions with no stability proofs, see e.g. Berg *et al.*[1] and Van Amerongen *et al.*[13]. Thus, a stronger theoretical understanding is required to avoid critical situations and guarantee performance of systems in the precence of magnitude and rate saturations.

In this paper, we show that when the input is subject to simultaneous magnitude and rate saturations, it is possible to obtain an attractive equilibrium using linear state feedback controllers*. Furthermore, the closed-loop systems domain of attraction can be made arbitrary large. We analyze the same problem for unstable systems and give some new results on the domain of attraction.

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*This problem has recently and independently been addressed by Lin[3]. The author analyzes the problem of stabilizing null-controllable systems magnitude and rate saturations. They propose linear control laws solving the state and output feedback control problem. The results differ from the results presented here in that the controllers require feedback from the actual input.

2 Problem Description

The problem we consider is stabilization of a single-input multiple-state system with simultaneous magnitude and rate saturations, i.e. systems which can be written in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\sigma_1(v) \quad (1)$$

$$\dot{v} = \sigma_2\left(\frac{1}{\tau}(u - v)\right) \quad (2)$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the commanded input, $v \in \mathbb{R}$ is an internal state of the input dynamics and $\sigma_1(v) \in \mathcal{U}_1$ is the actual input to the plant. The functions $\sigma_1 : \mathbb{R} \to \mathcal{U}_1$ and $\sigma_2 : \mathbb{R} \to \mathcal{U}_2$ are saturation functions defined as

$$\sigma_i(\cdot) \triangleq \operatorname{sign}(\cdot)\min(|\cdot|, \beta_i), \quad \beta_i > 0, \quad i = 1, 2. \quad (3)$$

which implies $\mathcal{U}_i = [-\beta_i, \beta_i]$, $i = 1, 2$. The input dynamics is shown in Figure 1.

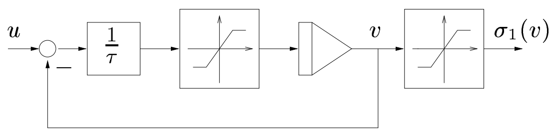

Figure 1: Input dynamics.

In this paper we consider the destabilizing effect of a rate limiter. Hence, we neglect the linear part of the input dynamics, and make the following assumption throughout the paper:

Assumption 1 Assume that τ in (2) is sufficiently "small", i.e. the linear part of the input dynamics can be neglected. △

Under Assumption1, (2) is called a pure rate saturation element possessing the following property:

Property 1 Let (2) satisfy Assumption 1, and let $v(T) = u(T)$ for some $T \geq t_0$. Then $v(t) = u(t)$, $t \geq T$ if $\dot{u}(t) \leq \beta_2$, $t \geq T$. △

The object is to derive a linear controller such that for any arbitrary large, bounded set of initial conditions, $\mathcal{X}$, the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is stable and $\lim_{t\to\infty} \|\boldsymbol{x}(t)\| = 0$. This problem is a semi-global stabilization problem and it is summarized in Definition 1, based on a definition given by Lin and Saberi[4].

Definition 1 (*Semi-Global Stability*) The equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is semi-globally asymptotically (exponentially) stabilizable by linear state feedback if, for any arbitrary large, bounded and a priori given sets of state initial conditions, $\mathcal{X} \subset \mathbb{R}^n$, there exists a linear control law, $u = \boldsymbol{g}^T\boldsymbol{x}$, such that the equilibrium of the closed-loop system is locally asymptotically (exponentially) stable and has $\mathcal{X}$ contained in its region of attraction. △

However, when the input nonlinearity has memory, like rate saturations, stability depends on the initial condition of the input. Hence, the equilibrium point may be unstable and attractive at the same time, i.e. an input initial condition may cause the state to leave the equilibrium point and then converge to it again as $t \to \infty$. Clearly, in this case stability is impossible to obtain, and the best possible result is an attractive equilibrium.

Now, we are ready to derive semi-global results for the full state feedback case. First, we consider a class of systems called null-controllable and show that for these systems there exists a linear control law such that the equilibrium is locally attractive with arbitrary large domain of attraction. Then, sufficient conditions for the existence of a linear controller for unstable systems are given.

3 Null-Controllable Systems

In this section, we consider a class of systems satisfying the following definition.

Definition 2 (*Null-controllability*) The system (1) is null-controllable with bounded controls if the eigenvalues of $\boldsymbol{A}$ are in the closed left half-plane and the pair $(\boldsymbol{A}, \boldsymbol{b})$ is stabilizable. △

To stabilize the system (1)–(2) we use linear control laws in the form

$$u = -\boldsymbol{b}^T\boldsymbol{P}(\gamma)\boldsymbol{x}, \quad (4)$$

where $\boldsymbol{P}(\gamma)$ is the solution of the Riccati equation

$$\boldsymbol{A}^T\boldsymbol{P}(\gamma) + \boldsymbol{P}(\gamma)\boldsymbol{A} - \boldsymbol{P}(\gamma)\boldsymbol{b}\boldsymbol{b}^T\boldsymbol{P}(\gamma) = -\boldsymbol{Q}(\gamma), \quad (5)$$

with $\boldsymbol{Q}(\gamma)$ a positive definite matrix $\forall\gamma \in (0, 1]$. The following proposition regarding $\boldsymbol{P}(\gamma)$ will be used throughout this section.

Proposition 1 Let the pair $(\boldsymbol{A}, \boldsymbol{b})$ be null-controllable and let $\boldsymbol{P}(\gamma)$ be the solution of (5) where $\boldsymbol{Q}(\gamma)$ is a positive definite matrix satisfying $\lim_{\gamma\to 0}\boldsymbol{Q}(\gamma) = \boldsymbol{0}$. Then,

$$\lim_{\gamma\to 0}\boldsymbol{P}(\gamma) = \boldsymbol{0}, \quad (6)$$

and $\boldsymbol{P}(\gamma)$ is positive definite $\forall\gamma \in (0, 1]$.

Proof: See Saberi *et al.*[6] and the references therein. △

We also need the following result for positive definite matrices in the proof of the main theorem of this section.

Proposition 2 Let $\boldsymbol{P}(\gamma)$ be a symmetric and positive definite matrix. Then, there exists a symmetric and positive definite matrix, $\boldsymbol{H}(\gamma)$, such that $\boldsymbol{P}(\gamma) = \boldsymbol{H}(\gamma)\boldsymbol{H}(\gamma)$.

Proof: See Strang[7]. △

Before stating the main theorem of this section, we need the following preliminary result.

Lemma 1 Let the system (1) be null-controllable and let the input dynamics (2) satisfy Assumption 1. Then, if $v(t_0) = u(t_0)$ the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is semi-globally exponentially stabilizable by linear state feedback.

Proof: To prove the theorem, we need to make some definitions. First, let $\mathcal{B}_M$ be a ball about the origin such that

$$\mathcal{X} \subseteq \mathcal{B}_M \stackrel{\Delta}{=} \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| \leq M\}, \tag{7}$$

and define a change of coordinates

$$\boldsymbol{z} \stackrel{\Delta}{=} \boldsymbol{H}(\gamma)\boldsymbol{x}. \tag{8}$$

where $\boldsymbol{P}(\gamma) = \boldsymbol{H}(\gamma)\boldsymbol{H}(\gamma)$. The existence of $\boldsymbol{H}(\gamma)$ is guaranteed by Proposition 2.

Next, consider the commanded input

$$u = \boldsymbol{b}^T\boldsymbol{P}(\gamma)\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{H}(\gamma)\boldsymbol{z}. \tag{9}$$

which give the following system matrix (when the input nonlinearities are neglected)

$$\boldsymbol{A}_c(\gamma) = \boldsymbol{H}(\gamma)\left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{b}^T\boldsymbol{P}(\gamma)\right)\boldsymbol{H}^{-1}(\gamma). \tag{10}$$

First, we want to pick a $\gamma \in (0,1]$ such that $u(t) \leq \beta_1$, $t \geq t_0$. To obtain this, let $0 < \gamma_1^* \leq 1$ be such that

$$\|\boldsymbol{b}^T\boldsymbol{H}(\gamma)\|\,\|\boldsymbol{H}(\gamma)\|M \leq \beta_1, \quad \forall\gamma \in \Gamma_1. \tag{11}$$

where $\Gamma_1 = (0, \gamma_1^*]$. Proposition 1 and the definition of $\boldsymbol{H}(\gamma)$, Proposition 2, guarantees the existence of γ_1^*. In addition, we want to guarantee that the time derivative of the commanded input never exceeds the rate limit, that is $\dot{u}(t) \leq \beta_2$, $t \geq t_0$. To do this, consider the time derivative of the input:

$$\dot{u} = -\boldsymbol{b}^T\boldsymbol{P}(\gamma)\dot{\boldsymbol{x}} = -\boldsymbol{b}^T\boldsymbol{P}(\gamma)\left(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u\right). \tag{12}$$

By assumption, $v(t_0) = u(t_0)$ and we want to have $v(t) = u(t)$, $t \geq t_0$. Consider:

$$\dot{u} = -\boldsymbol{b}^T\boldsymbol{P}(\gamma)\left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{b}^T\boldsymbol{P}(\gamma)\right)\boldsymbol{x}. \tag{13}$$

From (10) and (8), we get

$$\dot{u} = -\boldsymbol{b}^T\boldsymbol{H}(\gamma)\boldsymbol{A}_c(\gamma)\boldsymbol{z} \tag{14}$$

Now, let $\gamma_2^* \in (0,1]$ be such that

$$\|\boldsymbol{b}^T\boldsymbol{H}(\gamma)\boldsymbol{A}_c(\gamma)\|\,\|\boldsymbol{H}(\gamma)\|M \leq \beta_2, \quad \forall\gamma \in \Gamma_2. \tag{15}$$

where $\Gamma_2 = (0, \gamma_2^*]$ and γ_2^* exists, since

$$\lim_{\gamma \to 0} \boldsymbol{A}_c(\gamma) = \overline{\boldsymbol{A}}_c, \tag{16}$$

and it can be shown that $\|\overline{\boldsymbol{A}}_c\| < d$ (for a short proof, see Lin[3]). Then, let $\gamma^* = \min\{\gamma_1^*, \gamma_2^*\}$ and let c_M be a positive constant defined as

$$c_M \stackrel{\Delta}{=} \lambda_{max}(\boldsymbol{H}(\gamma^*))M, \tag{17}$$

where $\lambda_{max}(\boldsymbol{H}(\gamma^*))$ denotes the maximal (positive and real) eigenvalue of $\boldsymbol{H}(\gamma^*)$. Then, in the new coordinates, we have

$$\boldsymbol{z}(t_0) \in \mathcal{B}_{c_M} \stackrel{\Delta}{=} \{\boldsymbol{z} \in \mathbb{R}^n \mid \|\boldsymbol{z}\| \leq c_M\}. \tag{18}$$

since $\boldsymbol{z}^T(t_0)\boldsymbol{z}(t_0) = \boldsymbol{x}^T(t_0)\boldsymbol{P}(\gamma)\boldsymbol{x}(t_0) \leq c_M^2$. Pick a fixed $\gamma \in (0, \gamma^*]$ and consider the following Lyapunov function candidate

$$V = \boldsymbol{x}^T\boldsymbol{P}_\gamma\boldsymbol{x} = \boldsymbol{z}^T\boldsymbol{z}, \tag{19}$$

where $\boldsymbol{z} = \boldsymbol{H}_\gamma\boldsymbol{x}$ and $\boldsymbol{P}_\gamma = \boldsymbol{P}(\gamma)$ for a fixed $\gamma \in (0, \gamma^*]$. The time derivative of (19) becomes

$$\begin{aligned}\dot{V} = {} & \boldsymbol{z}^T\boldsymbol{H}_\gamma^{-1}\left(\boldsymbol{A}^T\boldsymbol{P}_\gamma + \boldsymbol{P}_\gamma\boldsymbol{A}\right)\boldsymbol{H}_\gamma^{-1}\boldsymbol{z} \\ & -2\boldsymbol{z}^T\boldsymbol{H}_\gamma\boldsymbol{b}\sigma_1(v).\end{aligned} \tag{20}$$

Now, observing that (15) implies that $\dot{u}(t) \leq \beta_2$, $t \geq t_0$ if $\boldsymbol{z}(t) \in \mathcal{B}_{c_M}$, $t \geq t_0$. Then, since $v(t_0) = u(t_0)$, Property 1 implies that $v(t) = u(t)$, $t \geq t_0$ and thus

$$\begin{aligned}\dot{V} = {} & \boldsymbol{z}^T\boldsymbol{H}_\gamma^{-1}\left(\boldsymbol{A}^T\boldsymbol{P}_\gamma + \boldsymbol{P}_\gamma\boldsymbol{A}\right)\boldsymbol{H}_\gamma^{-1}\boldsymbol{z} \\ & -2\boldsymbol{z}^T\boldsymbol{H}_\gamma\boldsymbol{b}\sigma_1\left(\boldsymbol{b}^T\boldsymbol{H}_\gamma\boldsymbol{z}\right), \quad \boldsymbol{z}(t) \in \mathcal{B}_{c_M}.\end{aligned} \tag{21}$$

and using (11) it follows that

$$\dot{V} \leq -\boldsymbol{z}^T\boldsymbol{H}_\gamma^{-1}\boldsymbol{Q}_\gamma\boldsymbol{H}_\gamma^{-1}\boldsymbol{z}, \quad \boldsymbol{z}(t) \in \mathcal{B}_{c_M}. \tag{22}$$

Since $\|\boldsymbol{z}(t)\|$ is strictly decreasing, it is clear that $\boldsymbol{z}(t_0) \in \mathcal{B}_{c_M}$ implies that $\boldsymbol{z}(t) \in \mathcal{B}_{c_M}$, $t \geq t_0$. Moreover, since $\boldsymbol{x}(t_0) \in \mathcal{B}_M$ implies that $\boldsymbol{z}(t_0) \in \mathcal{B}_{c_M}$ then (8) and (22) give

$$\dot{V} \leq -\boldsymbol{x}^T\boldsymbol{Q}_\gamma\boldsymbol{x}, \quad \boldsymbol{x}(t_0) \in \mathcal{X}. \tag{23}$$

Hence, from (19) and (23) it follows that for all $\boldsymbol{x}(t_0) \in \mathcal{X}$

$$\|\boldsymbol{x}(t)\| \leq k_M\|\boldsymbol{x}(t_0)\|e^{-\alpha_M(t-t_0)}, \tag{24}$$

which completes the proof. □

The preceding lemma requires that $v(t_0) = u(t_0)$, and this can, in general, not be satisfied, e.g. in the case of repeated steps in a reference signal. However, as the following theorem shows for the full state feedback case, an attractive equilibrium can be obtained for any bounded input initial condition.

Theorem 1 *(Null-Controllable Systems).* Let the system (1) be null-controllable and let the input dynamics (2) satisfy Assumption 1. Then, for any arbitrary large, bounded and a priori given set of state initial conditions, $\mathcal{X} \subset \mathbb{R}^n$, there exists a linear control law, $u = \boldsymbol{g}^T\boldsymbol{x}$, such that the equilibrium of the closed-loop system is locally attractive and has $\mathcal{X}$ contained in its region of attraction. Moreover, for $t \geq 2\beta_1/\beta_2$, $\boldsymbol{x}$ converges exponentially to the equilibrium.

Proof: First, notice that it is impossible to pick a $\gamma \in (0,1]$ which guarantee that $v(t) = u(t)$ for $t \geq t_0$ and for all $v(t_0)$. This is easily seen in the case when $\mathrm{sign}(v(t_0)) \neq \mathrm{sign}(u(t_0))$. Thus, in general, $v(t_0) \neq u(t_0)$ for all $\gamma \in (0,1]$.

Assume that we have picked a $\gamma \in (0,1]$ such that $|u(t)| \leq \beta_1$, $t \geq t_0$. Then, there exists a time T such that $v(T) = u(T)$. Since T is bounded, that is $T \leq 2\beta_1/\beta_2 < \infty$, there exists a ball $\mathcal{B}_N$ such that

$$\boldsymbol{x}(t_0) \in \mathcal{B}_M \Rightarrow \boldsymbol{x}(t) \in \mathcal{B}_N, \;\; t \in [t_0, t_0 + \frac{2\beta_1}{\beta_2}], \tag{25}$$

Then, Lemma 1 guarantees the existence of a linear control law $u = \boldsymbol{g}^T\boldsymbol{x}$ such that for all $\boldsymbol{x}(T) \in \mathcal{B}_N$

$$\|\boldsymbol{x}(t)\| \leq k_N \|\boldsymbol{x}(T)\| e^{-\alpha_N(t-T)}, \;\; t \geq T. \tag{26}$$

Moreover, from (25) we get that for all $\boldsymbol{x}(T) \in \mathcal{B}_N$ there exists some constants k_1 and k_2 such that

$$\|\boldsymbol{x}(t)\| \leq k_1 \|\boldsymbol{x}(t_0)\| + k_2, \;\; t \in [t_0, T], \tag{27}$$

and it is easy to verify that

$$\|\boldsymbol{x}(t)\| \leq (k_3 \|\boldsymbol{x}(t_0)\| + k_4) e^{-\alpha_N(t-t_0)}, \; t \geq t_0 \tag{28}$$

for some $k_3, k_4 > 0$ and $\forall\, \boldsymbol{x}(t_0) \in \mathcal{X}$. □

4 Unstable Systems

The results given in the previous section are not valid if the systems matrix has one or more eigenvalues with positive real part. This follows from the fact that

$$\lim_{\gamma \to 0} \boldsymbol{P}(\gamma) = \overline{\boldsymbol{P}}, \tag{29}$$

where $\overline{\boldsymbol{P}} \neq \boldsymbol{0}$. Assume that there exists a positive number $q \in \{1, 2, \cdots, (n-1)\}$ and a corresponding $r = n - q$ such that the system matrix (1) can be written in the form

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{bmatrix} \tag{30}$$

where $\boldsymbol{A}_{11} \in \mathbb{R}^{q \times q}$ has non-positive eigenvalues, $\boldsymbol{A}_{22} \in \mathbb{R}^{r \times r}$ has positive eigenvalues, and $\boldsymbol{A}_{12} \in \mathbb{R}^{q \times r}$. Then, the state can be divided into two parts,

$$\boldsymbol{x} = [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T]^T, \tag{31}$$

where $\boldsymbol{x}_1(t_0) \in \mathcal{X}_1 \subset \mathbb{R}^q$ and $\boldsymbol{x}_2(t_0) \in \mathcal{X}_2 \subset \mathbb{R}^r$, and we obtain the following system

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} \sigma(v), \tag{32}$$

where $\boldsymbol{x}_1 \in \mathbb{R}^q$, $\boldsymbol{x}_2 \in \mathbb{R}^r$ and with obvious dimensions and definitions of $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$.

Remark 1 Notice that if q does not exist, then the system is either null-controllable and thus $\boldsymbol{A}_{11} = \boldsymbol{A}$, $\boldsymbol{b}_1 = \boldsymbol{b}$ and $\boldsymbol{x}_1 = \boldsymbol{x}$, or it is not possible to obtain $\boldsymbol{A}_{21} \equiv \boldsymbol{0}$ and it follows that $\boldsymbol{A}_{22} = \boldsymbol{A}$, $\boldsymbol{b}_2 = \boldsymbol{b}$ and $\boldsymbol{x}_2 = \boldsymbol{x}$. △

Next, we make the following assumption on the system (32)

Assumption 2 Assume that (i) the pair $(\boldsymbol{A}, \boldsymbol{b})$ is stabilizable, (ii) the eigenvalues of $\boldsymbol{A}_{11}$ have non-positive real parts and (iii) the eigenvalues of $\boldsymbol{A}_{22}$ have positive real parts. △

Using Assumption 2, the following proposition can be stated.

Proposition 3 Let $\boldsymbol{P}(\gamma)$ be the solution of (5), where the pair $(\boldsymbol{A}, \boldsymbol{b})$ satisfy Assumption 2, and $\boldsymbol{Q}(\gamma)$ is a positive definite matrix satisfying $\lim_{\gamma \to 0} \boldsymbol{Q}(\gamma) = \boldsymbol{0}$. Then, $\boldsymbol{P}(\gamma)$ is positive definite $\forall \gamma \in (0,1]$ and

$$\lim_{\gamma \to 0} \boldsymbol{P}(\gamma) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\boldsymbol{P}}_{22} \end{bmatrix}. \tag{33}$$

Proof: Let $\overline{\boldsymbol{P}}_{22}$ be the solution of the Algebraic Riccati equation

$$\boldsymbol{A}_{22}^T \overline{\boldsymbol{P}}_{22} + \overline{\boldsymbol{P}}_{22} \boldsymbol{A}_{22} - \overline{\boldsymbol{P}}_{22} \boldsymbol{b}_2 \boldsymbol{b}_2^T \overline{\boldsymbol{P}}_{22} = \boldsymbol{0}, \tag{34}$$

Next, assume that there exists a $\delta > 0$ such that

$$\boldsymbol{Q}(\gamma_1) - \boldsymbol{Q}(\gamma_2) \leq \boldsymbol{0}, \; \forall\, 0 \leq \gamma_1 \leq \gamma_2 \leq \delta \tag{35}$$

Continuity at $\gamma = 0$ has been shown by Trentelman[12], i.e.

$$\lim_{\gamma \to 0} \boldsymbol{P}(\gamma) = \boldsymbol{P}(0). \tag{36}$$

Finally, it can be verified that (33) with $\overline{\boldsymbol{P}}_{22}$ given by (34) is $\boldsymbol{P}(0)$, and the proposition is proved. △

Next, define

$$\kappa_1 = \|\boldsymbol{b}_2^T \overline{\boldsymbol{H}}_{22}\| \, \|\overline{\boldsymbol{H}}_{22}\| \quad (37)$$

$$\kappa_2 = \|\boldsymbol{b}_2^T \overline{\boldsymbol{H}}_{22} \boldsymbol{A}_c(0)\| \, \|\overline{\boldsymbol{H}}_{22}\|, \quad (38)$$

and let $\mathcal{B}_{M_i}$, $i = 1, 2$, be two balls about the origin such that

$$\mathcal{X}_i \subseteq \mathcal{B}_{M_i} \triangleq \{\boldsymbol{x}_i \in \mathbb{R}^{n_i} \mid \|\boldsymbol{x}_i\| \leq M_i\}, \quad i = 1, 2 \quad (39)$$

where $n_1 = q$ and $n_2 = r$. Then the following lemma can be stated.

Lemma 2 Let the system (1)–(2) satisfy Assumptions 1 and 2. If $\kappa_1 M_2 < \beta_1$, $\kappa_2 M_2 < \beta_2$ and $v(t_0) = u(t_0)$, then for any arbitrary large and bounded $\mathcal{X}_1$ the equilibrium $\boldsymbol{x} = \boldsymbol{0}$ is exponentially stabilizable by linear state feedback.

Proof: Consider the state transformation $\boldsymbol{T} : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\boldsymbol{T} = \begin{bmatrix} \epsilon \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \quad (40)$$

Then defining $\overline{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x}$ we have

$$\dot{\overline{\boldsymbol{x}}} = \boldsymbol{A}(\epsilon)\overline{\boldsymbol{x}} + \boldsymbol{b}(\epsilon)\sigma_1(u) \quad (41)$$

where $\overline{\boldsymbol{x}}_1(t_0) \in \epsilon\mathcal{X}_1$, $\overline{\boldsymbol{x}}_2(t_0) \in \mathcal{X}_2$ and

$$\boldsymbol{A}(\epsilon) = \begin{bmatrix} \boldsymbol{A}_{11} & \epsilon\boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{bmatrix}, \quad \boldsymbol{b}(\epsilon) = \begin{bmatrix} \epsilon\boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}, \quad (42)$$

Then, since

$$\lim_{\epsilon \to 0} \|\overline{\boldsymbol{x}}(t_0)\| = \|\overline{\boldsymbol{x}}_2(t_0)\| \leq M_2 \quad (43)$$

there exists an ϵ^* such that

$$\kappa_1\|\overline{\boldsymbol{x}}(t_0)\| < \beta_1, \quad \kappa_2\|\overline{\boldsymbol{x}}(t_0)\| < \beta_2, \quad (44)$$

for all $\epsilon \in (0, \epsilon^*]$.

Next, let $\boldsymbol{P}(\epsilon, \gamma)$ denote the solution of (5) when $\boldsymbol{A}$ and $\boldsymbol{b}$ are replaced with $\boldsymbol{A}(\epsilon)$ and $\boldsymbol{b}(\epsilon)$. Since (34) is independent of ϵ it follows that

$$\lim_{\gamma \to 0} \boldsymbol{P}(\epsilon, \gamma) = \lim_{\gamma \to 0} \boldsymbol{P}(1, \gamma) = \overline{\boldsymbol{P}}, \quad (45)$$

where $\overline{\boldsymbol{P}}$ is given by (33). Hence, pick $\epsilon \in (0, \epsilon^*]$ such that (44) is satisfied and let $\overline{M}$ be such that $\|\overline{\boldsymbol{x}}(t_0)\| \leq \overline{M}$. Then, there exists $\gamma^* \in (0, 1]$ such that

$$\|\boldsymbol{b}^T(\epsilon)\boldsymbol{H}(\epsilon, \gamma)\| \, \|\boldsymbol{H}(\epsilon, \gamma)\|\overline{M} \leq \beta_1, \quad (46)$$

and

$$\|\boldsymbol{b}^T(\epsilon)\boldsymbol{H}(\epsilon, \gamma)\boldsymbol{A}_c(\epsilon, \gamma)\| \, \|\boldsymbol{H}(\epsilon, \gamma)\|\overline{M} \leq \beta_2. \quad (47)$$

for all $\gamma \in (0, \gamma^*]$. Then, the rest of the proof is similar to the proof of Lemma 1. □

The stability result for unstable systems is given in the following theorem.

Theorem 2 *(Unstable Systems)* Let the system (1)–(2) satisfy Assumptions 1 and 2. Furthermore, let $\mathcal{B}_{N_2}$ be such that $\boldsymbol{x}_2(t_0) \in \mathcal{X}_2$ implies that $\boldsymbol{x}_2(T) \in \mathcal{B}_{N_2}$, for any $T \leq 2\beta_1/\beta_2$. If $\kappa_1 N_2 < \beta_1$ and $\kappa_2 N_2 < \beta_2$, then for any arbitrary large, bounded and a priori given set of initial conditions $\mathcal{X}_1$, there exists a linear control law, $u = \boldsymbol{g}^T\boldsymbol{x}$, such that the equilibrium of the closed-loop system is locally attractive and has $\mathcal{X}_1 \times \mathcal{X}_2$ contained in its region of attraction. Moreover, for $t > 2\beta_1/\beta_2$, $\boldsymbol{x}$ has exponentially rate of convergence to the equilibrium.

Proof: The proof follows from the proofs of Theorem 1 and Lemma 2. □

Remark 2 Notice that this result is a generalization of the result for null-controllable systems since $\mathcal{X}_1$ can be arbitrary large and in the null-controllable case $\mathcal{X}_1 = \mathcal{X}$. △

5 Illustrative Examples

Two examples are used to illustrate the main results of this paper. First, a null-controllable system (integrator chain) is used to illustrate the destabilizing effect of the input initial condition, and secondly, a locally attractive equilibrium is shown for an unstable plant (F-16 fighter aircraft).

5.1 Integrator Chain

To illustrate the destabilizing effect of the input initial condition, consider a control system for three integrators with input magnitude and rate saturation, i.e.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sigma_1(v) \quad (48)$$

$$\dot{v} = \sigma_2\left(\frac{1}{\tau}(u - v)\right) \quad (49)$$

Further, let $\beta_1 = 2$, $\beta_2 = 1$ and $\mathcal{X} = \mathcal{B}_1$. It is well-known that for this system it is impossible to obtain global stability with linear control laws, and therefore we want to obtain local stability using a controller in the form $v = \boldsymbol{b}^T\boldsymbol{P}_\gamma\boldsymbol{x}$, where $\boldsymbol{P}_\gamma$ is the solution of (5) with

$$\boldsymbol{Q}_\gamma = \begin{bmatrix} \gamma^6 & 0 & 0 \\ 0 & \gamma^4 & 0 \\ 0 & 0 & \gamma^2 \end{bmatrix} \quad (50)$$

Then

$$\gamma_1 = 0.64 \leq \gamma_1^*, \quad \gamma_2 = 0.485 \leq \gamma_2^*. \quad (51)$$

Thus, picking $\gamma = 0.485$, the controller given by

$$u = -\begin{bmatrix} 0.1141 & 0.5679 & 1.1709 \end{bmatrix} \boldsymbol{x} \qquad (52)$$

results in a locally exponentially stable system when $u(t_0) = v(t_0)$, see Figure 2.

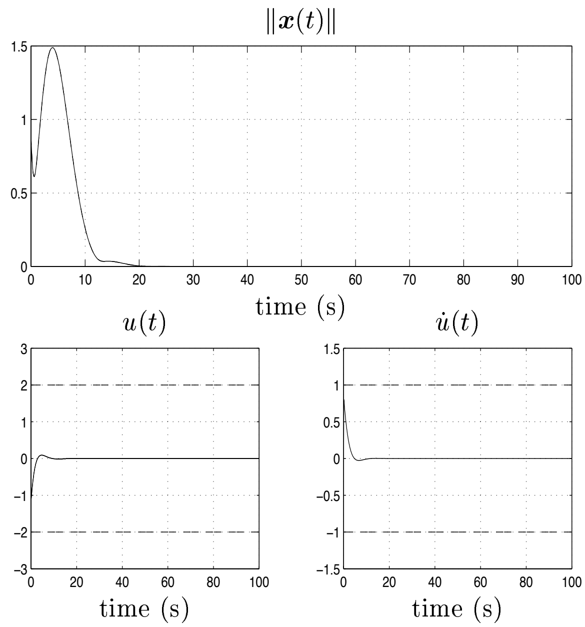

Figure 2: High gain linear controller when $v(0) = u(0)$.

However, the system may turn unstable when $u(t)$ has an initial condition different from the initial condition on $v(t)$. This is illustrated in Figure 3, where $v(t_0) = 2$. The initial values of $\boldsymbol{x}$ are not de-stabilizing, since $\|\boldsymbol{x}(t_0)\| = 1 \in \mathcal{B}_1$, but the initial value on $v(t)$ drives the system unstable.

Next, it can be shown that $\|\boldsymbol{x}(t)\| \leq 21$ for $t \leq 2\beta_1/\beta_2$. Using the same $\boldsymbol{Q}(\gamma)$ as above, it follows that $\gamma^* = 0.11$. This gives the linear controller

$$u = -\begin{bmatrix} 0.0013 & 0.0292 & 0.2656 \end{bmatrix} \boldsymbol{x}. \qquad (53)$$

The time response of this low gain controller are shown in Figure 4.

5.2 F-16 Control System

To illustrate the results for unstable systems, we use a model of the short period dynamics of an F-16 fighter aircraft. The model, taken from Miller and Pachter[5], is linearized at a flight condition of 10,000 feet in altitude and a speed of Mach 0.7. The state space model is

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -1.15 & 0.994 \\ 3.74 & -1.26 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} -0.177 \\ -19.5 \end{bmatrix} \delta, \qquad (54)$$

where q (rad/s) is the pitch rate, α (rad) is the angle of attack and δ (rad) is the input. The control objective is tracking of the pitch rate. Hence, we augment the system with an integral control state $z = \int_0^t q d\tau$. The magnitude and rate limits are 0.37 (rad) and 1 (rad/s) with a linear time constant, τ of 0.05. Thus, the overall system to be controlled is given by

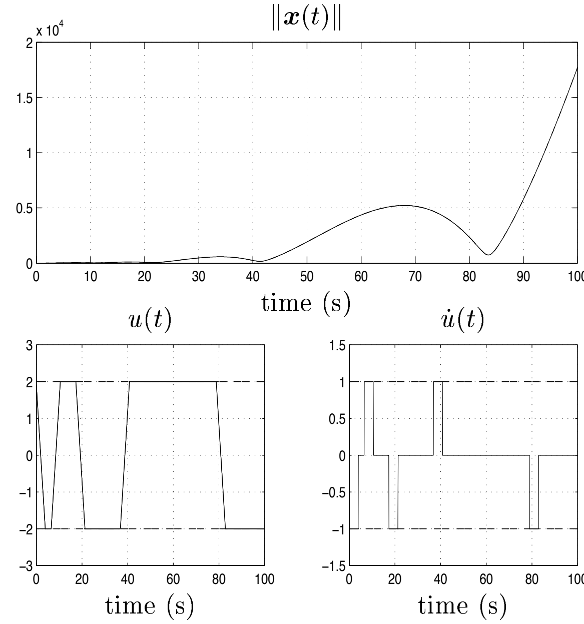

Figure 3: High gain linear controller when $v(0) = 2$.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1.15 & 0.994 & 0 \\ 3.74 & -1.26 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -0.177 \\ -19.5 \\ 0 \end{bmatrix} \sigma_1(v) \qquad (55)$$

$$\dot{v} = \sigma_2 \left(\frac{1}{0.05}(\delta_c - v) \right), \qquad (56)$$

where $\boldsymbol{x} = [\alpha, q, z]^T$ and δ_c is the commanded input.

The system is controllable, and to illustrate the destabilizing effect of saturating actuators consider the control law $\delta_c = -\boldsymbol{b}^T \boldsymbol{P}\boldsymbol{x}$ where

$$-\boldsymbol{b}^T \boldsymbol{P} = \begin{bmatrix} 0.1897 & 1.8288 & 5.4772 \end{bmatrix}, \qquad (57)$$

and $\boldsymbol{P}$ is the solution of (5) with

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 30 \end{bmatrix}. \qquad (58)$$

This controller was simulated with a sequence of steps in the reference and the result is shown in Figure 5. It is seen that the closed loop system is unstable due to the input limitations.

In order to derive a stabilizing controller for the system, consider the positive definite matrix

$$\boldsymbol{Q}(\gamma) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\gamma^4 & 0 \\ 0 & 0 & 30\gamma^2 \end{bmatrix}. \qquad (59)$$

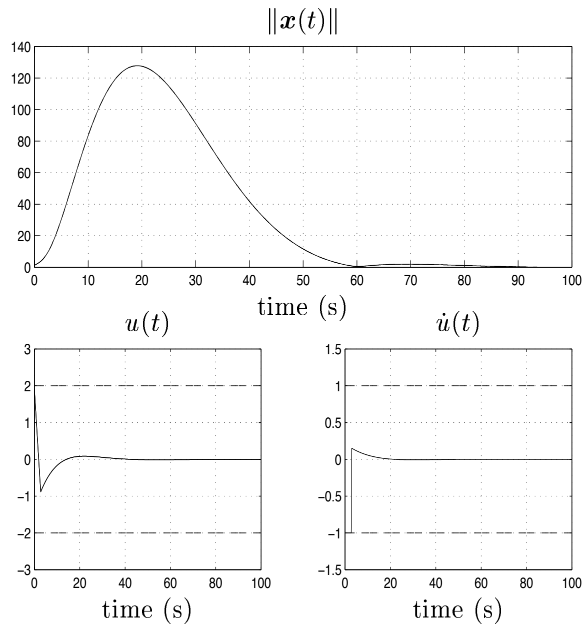

Figure 4: Low gain linear controller when $v(0) = 2$.

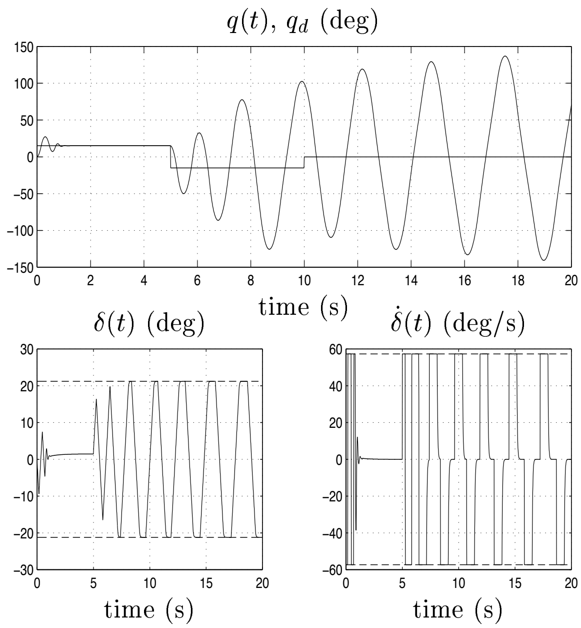

Figure 5: High gain linear controller.

If $\|\boldsymbol{x}(t)\| \leq 0.6$, for $t \leq 2\beta_1/\beta_2 = 0.74$, it can be shown that $\gamma = 0.24 \leq \gamma^*$ satisfy the conditions of Theorem 2. Using $\gamma = 0.24$, yields the following controller

$$\delta_c = \left[\begin{array}{ccc} 0.1766 & 0.1890 & 0.3155 \end{array}\right] \boldsymbol{x}. \qquad (60)$$

The simulation results are shown in Figure 6, and confirms that the equilibrium is locally attractive.

6 Conclusions

In this paper we have presented several semi-global results for linear systems with simultaneous magnitude and rate saturations on the input. It is shown that for null-controllable linear systems, a local attractive equilibrium can be obtained with linear control laws. Moreover, it is shown that the domain of attraction can be made arbitrary large. Unstable plants were also considered, and a sufficient condition for local stability was given. Full state feedback was assumed, and the output feedback case will be addressed in future publications.

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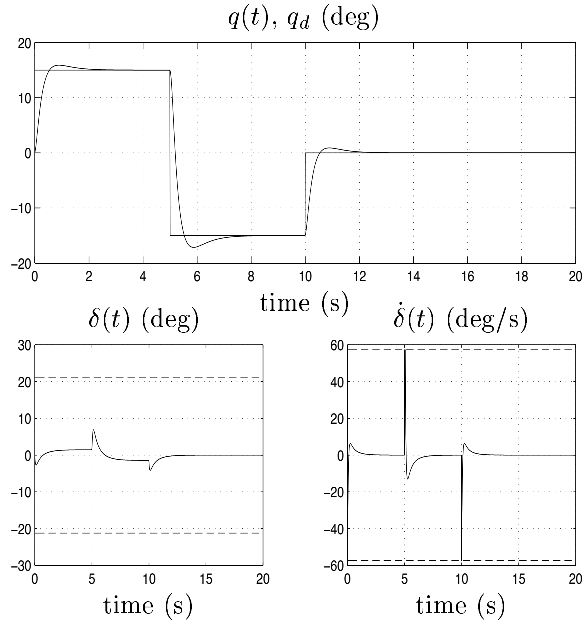

Figure 6: Low gain linear controller.

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