UGAS and ULES of Nonautonomous Systems: Applications to Integral Control of Ships and Manipulators

Antonio Loría† Thor I. Fossen* Andrew Teel†

Department of Engineering Cybernetics,
Norwegian Univ. of Sc. and Technology,
N-7034 Trondheim, Norway.
E-mail: tif@itk.ntnu.no.

Center for Control Engineering,
University of California,
Santa Barbara, CA 93106,
USA, E-mail: teel@ece.ucsb.edu
aloria@hamilton.ece.ucsb.edu.

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Abstract

Nonlinear, adaptive backstepping design is applied to the tracking control problem for a class of mechanical systems with constant disturbances. The adaptive algorithm provides integral action that guarantees zero steady-state tracking error. The main contribution of this paper is to show that the (time-varying) closed-loop tracking error system has an equilibrium, corresponding to zero steady-state tracking error, that is *uniformly* globally asymptotically stable (UGAS) and uniformly locally exponentially stable (ULES). These properties (and a uniform local Lipschitz condition) guarantee robustness of stability while weaker properties, like uniform global stability plus global convergence, do not.

Notation: $||\cdot||$ stands for the Euclidean norm of vectors and induced norm of matrices. $||\cdot||_\infty$ denotes the $\mathcal{L}_\infty$ norm. We denote by B_r the set $B_r \triangleq \{x \in \mathbb{R}^n : ||x|| \leq r\}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$, $\alpha \in \mathcal{K}$, if $\alpha(s)$ is strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if in addition $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\beta(s,t) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot,t) \in \mathcal{K}$ for each fixed $t \geq 0$ and $\beta(x,t) \to 0$ as $t \to \infty$ for each $x \geq 0$. Unless otherwise specified we use in general the letter c to denote a positive constant. For positive definite matrices we use the bounds $p_m I \leq P \leq p_M I$.

1 Introduction

When designing industrial control systems it is important to include integral action in the control law in order to compensate for slowly-varying and constant disturbances. This is necessary to avoid steady-state errors both in regulation and tracking. Often this is done in an *ad-hoc* manner with no proof of stability or convergence.

This paper discusses a method for integral action when backstepping. The integral part of the controller is provided by using adaptive backstepping ([6]) under the assumption of constant disturbances. The disturbances are estimated on-line by using parameter adaptation. The resulting error dynamics consists of the tracking error states and the parameter estimation error states. This leads to a new system of higher order than the original system. Moreover, the closed-loop system is typically time-varying, i.e., non-autonomous, since backstepping controllers typically do not yield linear error dynamics. The nonlinearities, which originally depended on the state of the system, must be rewritten in terms of the tracking error states and a time-dependent reference signal.

There are various types of asymptotic stability that can be pursued for time-varying nonlinear systems. The most useful of these, from a robustness point of view, are *uniform* (global) asymptotic stability and *uniform* (local) exponential stability. These are defined for the ordinary differential equation

$$\dot{x} = f(x,t) \qquad x(t_\circ) = x_\circ \tag{1}$$

as follows:

Definition 1 *The origin of the system (1) is said to be uniformly globally asymptotically stable (UGAS) if*

1. *there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $(x_\circ, t_\circ) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ and all $t \geq t_\circ$, we have*
$$||\phi(t,t_\circ,x_\circ)|| \leq \gamma(||x_\circ||) \; ; \tag{2}$$
2. *for each pair of strictly positive real numbers (r,σ) there exists a positive real number T such that*
$$||x_\circ|| \leq r \implies ||\phi(t,t_\circ,x_\circ)|| \leq \sigma \; \forall t \geq t_\circ + T \; . \tag{3}$$

Definition 2 *The origin of the system (1) is said to be uniformly locally exponentially stable (ULES) if there exist strictly positive real numbers r, k, λ such that*

$$||x_\circ|| \leq r \implies ||\phi(t, t_\circ, x_\circ|| \leq k||x_\circ||e^{-\lambda(t-t_\circ)} \ . \qquad (4)$$

The reason these types of stability are most useful is that (at least when $f(x,t)$ is locally Lipschitz in x uniformly in t) they guarantee total or robust stability. This is not necessarily the case for weaker forms of asymptotic stability for time-varying systems. To see this, consider a system of the form

$$\dot{x} = -a(t)x^3 \ . \qquad (5)$$

When $a(t)$ is such that the origin of this system is uniformly locally asymptotically stable and $a(t)$ is bounded, it is well-known (see [5, Lemma 5.4]) that the system

$$\dot{x} = -a(t)x^3 + d(t) \qquad (6)$$

is locally input-to-state stable; in particular, small bounded signals $d(t)$ and small initial conditions $x_\circ$ yield small state trajectories for all $t \geq 0$, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\max\{||x_\circ||, ||d||_\infty\} \leq \delta \implies ||\phi(t, t_\circ, x_\circ, d)|| \leq \epsilon \qquad (7)$$

for all $t \geq t_\circ \geq 0$. On the other hand, consider the case where $a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is continuous, strictly decreasing with $\lim_{t\to\infty} a(t) = 0$ but with $a(t)$ non integrable, i.e., $\lim_{t\to\infty} \int_0^\top a(\tau)d\tau = \infty$. In this case it is easy to see that the first part of the (UGAS) definition is satisfied for the system (5). Moreover, it can be shown that for all $(x_\circ, t_\circ) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ the trajectories of (5) satisfy $\lim_{t\to\infty} ||\phi(t, t_\circ, x_\circ)|| = 0$. Yet, for this $a(t)$, the system (6) is not locally input-to-state stable. [1]

In this paper we contribute a theorem that states sufficient conditions for UGAS and ULES for time-varying nonlinear systems in a form that is common in nonlinear adaptive control. Our sufficient conditions are stronger than typical persistency of excitation conditions in adaptive control (see, for example, [4]) but cover the problems that we are interested in for this paper. Our conditions admit a direct Lyapunov proof for UGAS and ULES instead of having to appeal to linear systems theory and the notion of "uniform complete observability". Besides, we have not been able to locate any results in the literature that convincingly prove *uniform* convergence, and hence UGAS, under persistency of excitation. Instead, the usual proofs simply establish trajectory by trajectory convergence (see[2] [7, Proposition 1]) or exponential convergence (for example, see[3] [4, Lemma A.5] or [5, Theorem 13.3]).

Recall that our stated reason for pursuing UGAS and ULES is robustness and/or input-to-state stability with respect to unmodeled disturbances when using certain controllers with integral action. It is worth noting that in [2] the authors solved a similar (global) input-to-*output* stability problem without appealing to UGAS by explicitly modeling the location of these "unmodeled disturbances" and further modifying the control law. In this case it was also assumed that the constant disturbances belonged to a known compact set.

The rest of our paper is organized as follows: in the next section we present our contribution on UGAS and ULES for a class of time-varying nonlinear systems. In section 3 a nonlinear ship tracking control problem is discussed where the tracking error dynamics is non-autonomous due to forward speed variations of the ship. Finally, UGAS and ULES is demonstrated for a robot manipulator in closed loop with a "Slotine and Li" [9] plus proportional-integral action controller.

2 Main Result

The analysis problem we will run into in this paper is to establish uniform global asymptotic stability (UGAS) and uniform local exponential stability (ULES) for a system in the form:

$$\dot{x}_1 = h(x_1, t) + G(x, t)x_2 \qquad (8a)$$

$$\dot{x}_2 = -PG(x, t)^\top \left(\frac{\partial W(x_1, t)}{\partial x_1}\right)^\top , \qquad (8b)$$

where $x_1 \in \mathbb{R}^m$, $x_2 \in \mathbb{R}^m$, $P = P^\top > 0$ and $W : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{C}^1$ function satisfying certain properties (see **A2** below). The assumptions in the following theorem will be used throughout the paper to establish UGAS and ULES when designing backstepping controllers with integral action, for mechanical systems.

Theorem 3 *If Assumptions* **A1** *and* **A2** *below hold, then the origin of the system (8) is UGAS.*

[1] Indeed, let $\epsilon > 0$ be given and suppose there exists a $\delta > 0$ such that (7) holds. Let $T > 0$ be such that $||a(t)x^3|| \leq 0.5\delta$ for all $||x|| \leq \epsilon$ and all $t \geq T$. Let $x_\circ = \delta$ and pick $d(t) = 0$ for all $t \in [0, T]$ and $d(t) = \min\{\delta, ||a(t)x^3(t) + 0.5\delta||\}$ for all $t > T$. By assumption $||x(t)|| \leq \epsilon$ for all $t \geq 0$. Also $x(t) \geq 0$ for all $t \geq 0$. Hence, by construction we have $\dot{x} = 0.5\delta$ for all $t > T$. But this contradicts $||x(t)|| \leq \epsilon$ for all $t \geq 0$.

[2] The cited proposition claims UGAS but the proof relies on a lemma that pays no attention to uniformity of convergence.

[3] Both of the cited results prove exponential convergence by associating to each trajectory a time-varying linear system that is shown to be ULES. No attention is paid to whether the convergence is uniform over the family of linear systems generated by all trajectories starting in a compact set. Only the result in [5] claims ULES.

A1 There exist continuous nondecreasing functions $\rho_j : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $(j = 1, 2, 3)$ such that, for all $t \geq 0$, $x \in \mathbb{R}^{n+m}$

$$\max\left\{ \|h(x_1,t)\|, \left\| \frac{\partial W(x_1,t)}{\partial x_1} \right\| \right\} \leq \rho_1(\|x_1\|)\,\|x_1\| \tag{9}$$

$$\|G(x,t)\| \leq \rho_2(\|x\|) \tag{10}$$

$$\max\left\{ \left\| \frac{\partial G(x,t)}{\partial t} \right\|, \left\| \frac{\partial G(x,t)}{\partial x_i} \right\| \right\} \leq \rho_3(\|x\|), \quad i = 1,\, 2\,. \tag{11}$$

Furthermore, for each compact set $K \subset \mathbb{R}^{n+m}$ there exist $g_m > 0$ such that

$$G(x,t)^\top G(x,t) \geq g_m^2 I \tag{12}$$

for all $(x,t) \in K \times \mathbb{R}_{\geq 0}$.

A2 There exist class-$\mathcal{K}_\infty$ functions α_1 and α_2 and a strictly positive real number $c > 0$ such that

$$\alpha_1(\|x_1\|) \leq W(x_1,t) \leq \alpha_2(\|x_1\|) \tag{13}$$

$$\frac{\partial W(x_1,t)}{\partial t} + \frac{\partial W(x_1,t)}{\partial x_1} h(x_1,t) \leq -c\,\|x_1\|^2\,. \tag{14}$$

Moreover, if $\alpha_2(s) \propto s^2$ then the origin is ULES. □

Proof. For uniform global stability we consider the Lyapunov function candidate $V_1 : \mathbb{R}^{n+m} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$V_1(x,t) = W(x,t) + \frac{1}{2} x_2^\top P^{-1} x_2 \tag{15}$$

which, from (15) and (13) is positive definite, radially unbounded and decrescent. The time derivative of $V_1(x,t)$ along the trajectories of (8), yields

$$\dot{V}_1(x,t) \leq -c\,\|x_1\|^2 \leq 0 \tag{16}$$

and therefore the origin is uniformly globally stable (UGS). That is, there exists a function γ of class $\mathcal{K}_\infty$ such that $\|x(t)\| \leq \gamma(\|x_\circ\|)$, $x_\circ = x(t_\circ)$.

Now we prove uniform attractivity. Notice that (8) can be rewritten as

$$\dot{x} = A(x,t)x + f(x_1,t) \tag{17}$$

where

$$A(x,t) = \begin{bmatrix} -I & G(x,t) \\ -G(x,t)^\top & 0 \end{bmatrix} \tag{18}$$

and

$$f(x_1,t) = \begin{bmatrix} h(x_1,t) + x_1 \\ -PG(x,t)^\top \left(\frac{\partial W(x_1,t)}{\partial x_1}\right)^\top + G(x,t)^\top x_1 \end{bmatrix}. \tag{19}$$

We will regard $f(x_1,t)$ as a perturbation to $\dot{x} = A(x,t)x$ and first prove that the system $\dot{x} = A(x,t)x$ is uniformly exponentially stable on each compact set of initial states. More precisely, we will show that for each $r > 0$ we can find a $\mathcal{C}^1$ function V_2 and strictly positive real numbers c_i $(i = 1,2,3)$ such that for all $(x,t) \in B_{\gamma(r)} \times \mathbb{R}_{\geq 0}$ we have $c_1\|x\|^2 \leq V_2(x,t) \leq c_2\|x\|^2$ and $\dot{V}_2 \leq -c_3\|x\|^2$. To that end we define $\omega \triangleq \gamma(r)$ and consider the Lyapunov function candidate $V_2 : B_\omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$V_2(x,t) = \frac{1}{2}\left(\|x_1\|^2 + \|x_2\|^2\right) - \varepsilon x_1^\top G(x,t) x_2 \tag{20}$$

where we impose $\varepsilon \leq 1/(2\bar{\rho}_2)$ with $\bar{\rho}_2 = \rho_2(\omega)$, so that $0.25\|x\|^2 \leq V_2(t,x) \leq 0.75\|x\|^2$ for all $(x,t) \in B_\omega \times \mathbb{R}_{\geq 0}$. The time derivative of $V_2(x,t)$ along the trajectories of $\dot{x} = A(x,t)x$ is

$$\begin{aligned} \dot{V}_2(x,t) &= -\|x_1\|^2 - \varepsilon x_1^\top \dot{G}(x,t) x_2 - \varepsilon x_1^\top G(x,t) x_2 \\ &\quad + \varepsilon x_1^\top G(x,t) G(x,t)^\top x_1 - \varepsilon x_2^\top G(x,t)^\top G(x,t) x_2\,. \end{aligned}$$

From (10) and (11) there exists $g'_M > 0$ such that $\|\dot{G}(x,t)\| \leq g'_M$ for all $(x,t) \in B_\omega \times \mathbb{R}_{\geq 0}$. Hence after some straightforward boundings we obtain that $\dot{V}_2(x,t)$ satisfies

$$\begin{aligned} \dot{V}_2(x,t) &\leq -\left(\frac{1}{2} - \varepsilon\bar{\rho}_2^2\right)\|x_1\|^2 - \frac{\varepsilon g_m^2}{2}\|x_2\|^2 - \frac{1}{4}\|x_1\|^2 \\ &\quad - \frac{1}{2}\begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix}^\top \begin{bmatrix} 1/2 & -\varepsilon(g'_M + \bar{\rho}_2) \\ -\varepsilon(g'_M + \bar{\rho}_2) & \varepsilon g_m^2 \end{bmatrix} \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix} \end{aligned}$$

for all $(x,t) \in B_\omega \times \mathbb{R}_{\geq 0}$. Clearly, if

$$\varepsilon \leq \frac{1}{2}\min\left\{ \frac{1}{\bar{\rho}_2^2}, \frac{g_m^2}{(g'_M + \bar{\rho}_1\bar{\rho}_2)^2} \right\} \tag{21}$$

then

$$\dot{V}_2(x,t) \leq -c_3\|x\|^2 \tag{22}$$

with $c_3 = 0.25\min\{\varepsilon g_m^2, 1\}$.

Using the calculations above we proceed to prove global uniform attractivity for the perturbed system (17). Consider the Lyapunov function candidate $\mathcal{V} : B_\omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $\mathcal{V}(x,t) \triangleq \mu V_1(x,t) + V_2(x,t)$ where $\mu > 0$ is to be specified later. From (15), (20) and (13) it follows that

$$\frac{1}{4}\|x\|^2 \leq \mathcal{V}(x,t) \leq \alpha_3(\|x\|) \tag{23}$$

where we defined $\alpha_3(s) \triangleq \mu\alpha_2(s) + \frac{\mu}{p_m}s^2 + 0.75s^2$ and we remind the reader that $P \geq p_m I > 0$. From (16), (17) and (22), we have

$$\dot{\mathcal{V}}(x,t) \leq -c\mu\|x_1\|^2 - c_3\|x\|^2 + \left\| \frac{\partial V_2}{\partial x}(x,t) \right\| \|f(x,t)\| \tag{24}$$

while from Assumption **A1**, there exist continuous and nondecreasing functions $\rho_4(||x||)$ and $\rho_5(||x||)$ such that

$$||f(x_1,t)|| \leq \rho_4(||x||)\,||x_1|| \quad (25)$$

$$\left\|\frac{\partial V_2}{\partial x}(x,t)\right\| \leq \rho_5(||x||)\,||x|| \ , \quad (26)$$

using these bounds in (24) we obtain

$$\dot{\mathcal{V}}(x,t) \leq -\left(c\mu - \frac{(\bar{\rho}_4\bar{\rho}_5)^2}{2c_3}\right)||x_1||^2 - \frac{c_3}{2}||x||^2 \quad (27)$$

where $\bar{\rho}_i = \rho(\omega)$, $i = 4, 5$ that is, $\dot{\mathcal{V}}(x,t)$ is negative definite on $B_\omega \times \mathbb{R}_{\geq 0}$ if μ is sufficiently large so that $\mu \geq (\bar{\rho}_4\bar{\rho}_5)^2/(2cc_3)$. From the last term on the right hand side of (27) we obtain that $\dot{\mathcal{V}}(t) \leq -\alpha_4(\mathcal{V}(t))$ for all t such that $x(t) \in B_\omega$ and where we have defined $\alpha_4(s) \triangleq -0.5c_3[\alpha_3^{-1}(s)]^2$ which is clearly of class $\mathcal{K}_\infty$. Moreover, since $x(t) \in B_\omega$ for all $(x_\circ, t_\circ) \in B_r \times \mathbb{R}_{\geq 0}$, it follows that $\dot{\mathcal{V}}(t) \leq -\alpha_4(\mathcal{V}(t))$ for all $t \geq 0$ therefore, from standard comparison theorems [5, Lemma 3.4] we obtain the existence of a function $\beta \in \mathcal{KL}$ such that $\mathcal{V}(t) \leq \beta(\mathcal{V}_\circ, t - t_\circ)$ – where $\mathcal{V}_\circ \triangleq \mathcal{V}(x_\circ, t_\circ)$ – which implies, from (23), that $||x(t)|| \leq 2\sqrt{\beta(\alpha_3(||x_\circ||), t - t_\circ)}$ for all $(x_\circ, t_\circ) \in B_r \times \mathbb{R}_{\geq 0}$. Therefore, given $r > 0$ and $\sigma > 0$ if we pick T such that $2\sqrt{\beta(\alpha_3(r), T)} \leq \sigma$, then (3) holds.

To complete the proof, notice that if $\alpha_2 \propto s^2$ then so is α_3 and ULES follows from standard results (see e.g. [5, Corollary 3.4]), looking at (23) and (27). ■

Remark 4 It is clear from the method of proof that this result hinges upon the ability to show that the unperturbed system $\dot{x} = A(x,t)x$ is ULES. The decomposition into a ULES and a perturbation is inspired by the results in [11]. For ULES of $\dot{x} = A(x,t)x$ we have imposed the bound (12). Even though for the applications considered in this paper this condition suffices to show UGAS, for other problems it may be too restrictive. In [8] we have established UGAS and ULES with (12) replaced by a uniform persistency of excitation condition.

3 Ship control

When designing ship control systems, integral action is needed in order to compensate for constant (or slowly-varying) environmental disturbances[4] due to

- slowly-varying ocean currents
- 2nd-order wave-induced drift forces
- slowly-varying wind forces.

[4]See [1] for details.

3.1 Ship model

Figure 1: Three degrees-of-freedom vessel.

We will consider a nonlinear supply vessel in 3 degrees of freedom, see Figure **??**. The example is taken from Godhavn et *al.* [3]. The state-space model is written as:

$$\dot{\eta} = R(\psi)\nu \quad (28)$$

$$M(U)\dot{\nu} + n(\nu, U) = \tau + R^\top(\psi)e. \quad (29)$$

$$\dot{e} = 0 \quad (30)$$

where $\eta = [x, y, \psi]^\top \in \mathbb{R}^3$ is the position/heading vector decomposed in the earth-fixed reference frame, $\nu = [u, v, r]^\top \in \mathbb{R}^3$ is the velocity vector decomposed in the body-fixed reference frame, $U(t) \in \mathbb{R}$ is the forward speed of the ship, $M(U) \in \mathbb{R}^{3\times 3}$ is the inertia matrix, $n(\nu, U) \in \mathbb{R}^3$is a nonlinear function of Coriolis, centripetal and damping forces, $e \in \mathbb{R}^3$ is the environmental disturbances to be compensated for by integral action and

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (31)$$

is the rotation matrix in yaw. It is clear that this matrix is orthogonal, i.e. $R^\top R = I$.

3.2 Backstepping with integral action

The control objective of the supply vessel is tracking of a smooth time-varying reference trajectory $\eta_d \in C^r$. In [3] an adaptive backstepping tracking control law for the nonlinear system (28)–(30) was proposed and global *convergence* of all state trajectories was proven. It is not our purpose to repeat the backstepping derivations for

the controller of [3] but using Theorem 3, we provide a simple proof of UGAS and ULES of the closed loop.

For the system (28)–(30) let us define the error signal $z_1 = \eta - \eta_d$, hence the error velocity is $\dot{z}_1 = R\nu - \dot{\eta}_d$. Also let us introduce the variable $z_2 = C_1 z_1 + (R\nu - \dot{\eta})$, where $C_1 > 0$. With this notation, the following adaptive controller was proposed in [3]

$$\begin{aligned} \tau &= n(\nu, U) - R^\top \hat{e} - M(U) R^\top (\dot{R}\nu - \ddot{\eta}_d) \\ &\quad - M(U) R^\top (C_1 \dot{z}_1 + z_1 + C_2 z_2) && (32a) \\ \dot{\hat{e}} &= \Gamma R M^{-T}(U) R^\top z_2 && (32b) \end{aligned}$$

where $C_2 > 0$. The weight matrix $\Gamma = \Gamma^\top > 0$ satisfies the Lyapunov function

$$V(z, \tilde{e}) = \frac{1}{2}(z_1^\top z_1 + z_2^\top z_2 + \tilde{e}^\top \Gamma^{-1} \tilde{e}) \tag{33}$$

where $\tilde{e} = \hat{e} - e$. The time derivative of $V(z, \tilde{e})$ along the trajectories of the closed loop (28)–(30), (32) yields

$$\dot{V}(z, \tilde{e}) = -z_1^\top C_1 z_1 - z_2^\top C_2 z_2 \,. \tag{34}$$

Clearly, $\dot{V}(z, \tilde{e})$ is negative *semi*-definite. In addition, the resulting error dynamics becomes non-autonomous due to the time dependence on $U(t)$. To that end, as it was done in [3], global convergence of $z(t)$ and global boundedness of $\tilde{e}$ can be concluded using standard arguments and Barbalat's Lemma. In contrast to this, based on Theorem 3, a closer look at the closed loop system leads us to the proof of UGAS. First, the error dynamics is

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} -C_1 & I \\ -I & -C_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ -RM^{-1}(U)R^\top \end{bmatrix} \tilde{e} && (35a) \\ \dot{\tilde{e}} &= -\Gamma R M^{-T}(U) R^\top z_2 \,. && (35b) \end{aligned}$$

To invoke Theorem 3 note that the closed loop system (35) is of the form (8) with $x_1 = \mathrm{col}(z_1, z_2)$, $x_2 = \tilde{e}$, $G(x_1, t)^\top = [0, -R^\top M^{-\top}(U) R]$, $P = \Gamma$. Moreover since $R(\psi)$ is an orthogonal bounded transformation and $M(U)$ is also uniformly bounded, $G(x_1, t)$ satisfies (10) and (11) with constant ρ_2 and ρ_3. Also note that (9) holds with $\rho_1 = \text{const}$. Finally, from (33) and (34) it follows that Assumption **A2** holds with $W(x_1, t) = 0.5\,\|z\|^2$. Therefore, the ship control system (35) is UGAS and ULES.

4 Robot tracking control with integral action

Consider the nonlinear robot model

$$\begin{aligned} \dot{q} &= v && (36a) \\ M(q)\dot{v} + C(q, v)v + g(q) &= \tau && (36b) \end{aligned}$$

where $M(q) = M^\top(q) > 0$ is the inertia matrix, $C(q, v)$ is a matrix of Coriolis and centripetal terms which satisfies the skew-symmetry property: $\xi^\top(0.5\dot{M}(q) - C(q, v))\xi = 0$, $\forall \xi \in \mathbb{R}^n$ and $g(q)$ is a vector of gravitational forces and moments. $q \in R^n$ is a vector of joint angles, $v \in R^n$ is a vector of joint angular rates and $\tau \in R^n$ is a vector of control torques.

We revisit the well known algorithm of Slotine and Li [9] and show that UGAS and ULES is still possible when integral action is added to the control law. For the desired reference trajectory $q_d \in \mathcal{C}^1$ and reference velocity $v_d = \dot{q}_d$ we define the *virtual reference*

$$v_r = v_d - \Lambda \tilde{q} \tag{37}$$

where $\Lambda > 0$ is a diagonal design matrix and $\tilde{q} = q - q_d$ is the tracking error. From (37) we have that

$$\tilde{v} = -\Lambda \tilde{q} + s \tag{38}$$

where $\dot{\tilde{q}} = \tilde{v}$. As it is well known now [10] the following control law

$$\tau = M(q)\dot{v}_r + C(q, v)v_r + g(q) - K_d s - K_q \tilde{q} \tag{39}$$

where $K_d = K_d^\top > 0$ and $K_q = K_q^\top > 0$ are design matrices, renders the robot system (36) globally exponentially stable. Indeed, GES of the closed loop system (36), (39):

$$\begin{bmatrix} \dot{\tilde{q}} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} -\Lambda & I \\ -M(q)^{-1}K_p & M(q)^{-1}(C(q, v) + K_d) \end{bmatrix} \begin{bmatrix} \tilde{q} \\ s \end{bmatrix} \tag{40}$$

is easily proven by means of the Lyapunov function candidate [10]

$$V(s, \tilde{q}, q_d(t)) = \frac{1}{2}(s^\top M(q) s + \tilde{q}^\top K_q \tilde{q}) \tag{41}$$

which is uniformly positive definite and radially unbounded in $(s, \tilde{q})$. The time derivative of $V(s, \tilde{q})$ along (40) yields

$$\dot{V}(s, \tilde{q}, q_d(t)) = -s^\top K_d s - \tilde{q}^\top K_q \Lambda \tilde{q} \tag{42}$$

which is negative definite in $(s, \tilde{q})$.

Even though GES can be proven under the assumption that all the robot parameters are known, it is of common practice to add an integral action to the control law in order to correct steady state errors which may be due to a mismatch between the true parameters of the robot model and those actually used in the feedback law. Therefore, instead of (39) it is desirable to use the controller

$$\begin{aligned} \tau &= M(q)\dot{v}_r + C(q, v)v_r + g(q) - K_d s - K_q \tilde{q} + \nu && (43a) \\ \dot{\nu} &= -K_i s && (43b) \end{aligned}$$

where K_i is a positive definite matrix. The closed loop of (36) with (43) yields

$$\begin{bmatrix} \dot{\tilde{q}} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} -\Lambda & I \\ -M(q)^{-1}K_p & M(q)^{-1}(C(q,v)+K_d) \end{bmatrix} \begin{bmatrix} \tilde{q} \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ M(q)^{-1} \end{bmatrix} \nu \tag{44a}$$

$$\dot{\nu} = -K_i s \ . \tag{44b}$$

Note that the closed loop system (44) is of the form (8) with $x_1 = \text{col}(\tilde{q}, s)$, $x_2 = \nu$, $G(x_1,t)^\top = [0, M^{-\top}(\tilde{q} + q_d(t))]$, $P = K_i$ and $W(x_1,t) = V(s,\tilde{q},q_d(t))$ as defined in (41). Since $q_d(t) \in \mathcal{C}^1$, $G(x_1,t)$ satisfies (10) and (11) while (9) holds with $\rho_1 = \text{const}$. Finally, Assumption **A2** is clearly satisfied from (41) and (42). Therefore, the robot control system (44) is UGAS and ULES.

5 Conclusions

In this paper UGAS and ULES of nonlinear non-autonomous systems where constant disturbances are compensated for by using an integral controller have been discussed. Integral action is provided by using adaptive backstepping. The main result of the paper is a theorem for UGAS/ULES which is intended as a design tool when designing industrial controllers with integral action. Emphasis is placed on non-autonomous systems where convergence cannot be determined by using Krasovskii-LaSalle's invariance theorem. Moreover, for time-varying systems, arguments based on Barbalat's Lemma as often used in the Robotics literature, do not lead to uniform convergence. It has been shown that convergence is important for robustness. Applications to ship and robot control have been used to demonstrate the utility of our theorem in establishing UGAS of control systems with integral action.

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