Global output tracking control of a class of Euler-Lagrange systems with monotonic nonlinearities in the velocities

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Abstract

In this paper, we address the problem of output feedback tracking control of a class of Euler-Lagrange systems subject to nonlinear dissipative loads. By imposing a monotone damping condition on the nonlinearities of the unmeasured states, the common restriction that the nonlinearities be globally Lipschitz is removed. The proposed observer-controller scheme renders the origin of the error dynamics uniformly globally asymptotically stable, in the general case. Under certain additional assumptions, the result continue to hold for a simplified control law that is less sensitive to noise and unmodeled phenomena.

1 Introduction

In contrast to the well developed theory of linear systems, results for nonlinear systems do not apply in general, but are restricted to classes of systems for which certain structural properties can be exploited. For instance, when only a subset of the states are available for measurement, simple algebraic tests are available for linear systems, that, if confirmative, ensure that the problem of output feedback tracking is solvable by means of an observer in conjunction with a state feedback control law. Moreover, the observer and the control law can be designed independently, a fact which is referred to as the *separation principle* for linear systems. For nonlinear systems in general, even an estimate that converges exponentially fast to the actual state, does not guarantee stability of the closed loop system when used in a state feedback control law.

Among the first papers on observer design for systems with nonlinearities in the unmeasured states was one by Thau (1973), whose results were later generalized by Kou et al. (1975). They considered systems in the form

$$\begin{aligned} \dot{x} &= Ax + g(t,u,y) + f(t,u,x) \\ y &= Cx \end{aligned}$$

where f is assumed to be globally Lipschitz in x with Lipschitz constant γ. They went on to construct an observer as follows

$$\dot{\hat{x}} = A\hat{x} + g(t,u,y) + f(t,u,\hat{x}) + L(y - C\hat{x})$$

where L is the observer gain matrix. Their main result stated that the estimate $\hat{x}$ will converge to the true state x, provided that $\gamma < \lambda_{\min}(Q)/2\lambda_{\max}(P)$, where P and Q are symmetric positive definite matrices satisfying the Lyapunov equation $(A-LC)^T P + P(A-LC) = -Q$. Raghavan and Hedrick (1994), and later Rajamani

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(1998), provided systematic approaches to finding an observer gain matrix L that satisfies this condition. In two recent papers, Arcak and Kokotović (1999a, 1999b) remove the global Lipschitz condition on f, and instead require a monotonic damping property to hold. As we will demonstrate, this damping property is of interest in offshore applications.

The papers by Arcak and Kokotović, and a recent paper by Loria and Panteley (1999), are the main sources of motivation for this work. Loria and Panteley (1999) present a separation principle for fully actuated Euler-Lagrange systems that can be transformed, via a global change of coordinates, into the following form

$$\dot{q} = J(q)\nu \tag{1}$$

$$M\dot{\nu} + v(q) = \tau \tag{2}$$

where q and ν are n-dimensional vectors of generalized positions and velocities, respectively, M is a symmetric positive definite and constant matrix, τ is the control input, and J is an $n \times n$ matrix as a function of q. In contrast to the numerous references made to ship dynamics in Loria and Panteley (1999), the class of systems described by equations (1) and (2) does not include systems where $\dot{\nu}$ depends on ν. Clearly, this fact excludes offshore structures and ships from the class, since they will be subject to coriolis and centripetal forces, as well as hydrodynamic drag. In general, these terms are nonlinear functions of the velocities. As already mentioned, nonlinearities in the unmeasured states are not trivial to include in the analysis, and the global output feedback tracking control problem has not been solved for general Euler-Lagrange systems (the global set-point regulation problem has been solved, though (Berghuis and Nijmeijer 1993)). In the special case of one degree-of-freedom systems, Loria (1996) presented the first explicit global solution to the problem. In Besançon (1998), a more elegant solution to the one degree-of-freedom problem was presented. For higher order systems, a semi-global result has been achieved by Loria and Nijmeijer (1998).

We extend the class of systems considered in Loria and Panteley (1999) by adding a term, nonlinear in the unmeasured states, that satisfies the monotone damping property employed for observer design in Arcak and Kokotović (1999b). Although this property allows the unmeasured states to be raised to any power, it ensures that the *unboundedness observability* property of Mazenc et al. (1994) holds. The observer-controller scheme proposed in this paper, renders the origin of the error dynamics uniformly globally asymptotically stable (UGAS). In a separate result, positive definiteness of the linear damping term, or alternatively, an additional assumption on the matrix $J(q)$, is exploited to obtain UGAS by means of a simplified control law.

2 Notation and definitions

We will denote the set of natural numbers by $\mathbb{N}$, the Euclidean space of dimension n by $\mathbb{R}^n$, the set of nonnegative real numbers by $\mathbb{R}_+$, and the *closed* ball of radius r by B_r. The 2-norm of elements of $\mathbb{R}^n$, as well as the induced matrix 2-norm, is denoted by $\|\cdot\|$. Estimates will be denoted by a hat, that is, $\hat{x}$ and $\hat{\dot{x}}$ are estimates of x and $\dot{x}$, respectively. The following definitions are taken from Khalil (1996) and Krstić et al. (1995).

Definition 1 *A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. If, in addition, $\alpha(r) \to \infty$ as $r \to \infty$ it is said to be a class $\mathcal{K}_\infty$ function. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class $\mathcal{KL}$ function if, for each fixed s, the mapping $\beta(r, s)$ is a class $\mathcal{K}$ function with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

Definition 2 *The solution $x(t; x_0, t_0)$ of $\dot{x} = f(t, x)$ is uniformly globally bounded (UGB) if for each $x_0 \in \mathbb{R}^n$ there exists a constant b (independent of t_0), such that*

$$\|x(t)\| \leq b(x_0)$$

Definition 3 *The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is uniformly globally asymptotically stable (UGAS) if*

1. *it is uniformly globally stable (UGS), that is, there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ and all $t \geq t_0$, we have*
$$\|x(t)\| \leq \gamma(\|x_0\|);$$

2. *for each pair of strictly positive real numbers ϵ and r, there exists a positive real number T such that*
$$\|x(t)\| \leq \epsilon, \quad \forall t \geq t_0 + T(\epsilon, r), \ \forall \|x(t_0)\| < r \tag{3}$$

Definition 4 *The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is uniformly locally exponentially stable (ULES) if there exist positive constants k, γ and c such that*

$$\|x(t)\| \leq k \|x(t_0)\| e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0, \ \forall \|x(t_0)\| < c \tag{4}$$

If (4) is satisfied for any initial state $x(t_0)$, then the equilibrium point is uniformly globally exponentially stable (UGES).

Definition 5 *The system $\dot{x} = f(t, x, u)$ is input-to-state stable (ISS) if there exist a class $\mathcal{KL}$ function β, and a class $\mathcal{K}$ function γ, such that for any initial state $x(t_0)$, and any bounded input $u(t)$, the solution $x(t)$ exists and satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} u(\tau)\right)$$

3 Problem statement

In the Lyapunov analysis that follows, we will require a monotone damping property to hold for the function of the unmeasured states (denoted d in (8) below). More precisely, we require that the function $d : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and satisfies

$$(x - y)^T P(d(x) - d(y)) \geq 0, \quad \forall x, y \in \mathbb{R}^n \tag{5}$$

where $P = P^T > 0$. For the special case when d is continuously differentiable, the following lemma provides a test for this property (see Ortega and Rheinboldt 1970).

Lemma 1 (monotone damping property) *Suppose $d : \mathbb{R}^n \to \mathbb{R}^n$ satisfies*

$$P\left[\frac{\partial d}{\partial x}\right] + \left[\frac{\partial d}{\partial x}\right]^T P \geq 0, \quad \forall x \in \mathbb{R}^n \tag{6}$$

where $P = P^T > 0$. Then, (5) holds for d.

We consider systems in the following form

$$\dot{q} = J(q)\nu \tag{7}$$

$$M\dot{\nu} + D\nu + d(\nu) + v(q) = \tau \tag{8}$$

where $q, \nu \in \mathbb{R}^n$, M and D are constant matrices satisfying $M = M^T > 0$ and $D + D^T \geq 0$, respectively, and $d(\nu)$ satisfies (5) with $P = I$. As in Loria and Panteley (1999) we will assume that $J : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ has the following properties

Property 1 *$J(q)$ is invertible and satisfies $0 < k_j \leq \|J(q)\| \leq k_J$ for all $q \in \mathbb{R}^n$.*

Property 2 *$\frac{d}{dt}J(q) = \dot{J}(q, \dot{q})$ is globally Lipschitz in $\dot{q}$, uniformly in q, with Lipschitz constant L_j.*

Problem 1 (the global output feedback tracking control problem) *Consider the system given by equations (7)–(8). Let a desired trajectory, $q_d : [t_0, \infty) \to \mathbb{R}^n$, be given, that has continuous first and second derivatives whose norms are bounded above by β_d. Find a controller*

$$\begin{aligned} \dot{\zeta} &= f(t, \zeta, q) \\ \tau &= h(t, \zeta, q) \end{aligned}$$

such that $q \to q_d$, and $\dot{q} \to \dot{q}_d$, as $t \to \infty$.

4 Main results

4.1 Observer design

Copying equations (7)–(8) and adding output injection terms yield the following observer

$$\begin{aligned} \dot{\hat{q}} &= J(q)\hat{\nu} + K_{o_1}\tilde{q} && (9) \\ M\dot{\hat{\nu}} + D\hat{\nu} + d(\hat{\nu}) + v(q) &= \tau + MK_{o_2}(q)\tilde{q} && (10) \end{aligned}$$

with error dynamics, in terms of $\tilde{q} \triangleq q - \hat{q}$ and $\tilde{\nu} \triangleq \nu - \hat{\nu}$, as follows

$$\begin{aligned} \dot{\tilde{q}} &= -K_{o_1}\tilde{q} + J(q)\tilde{\nu} && (11) \\ \dot{\tilde{\nu}} &= -K_{o_2}(q)\tilde{q} - M^{-1}\left[D\tilde{\nu} + d(\nu) - d(\hat{\nu})\right] && (12) \end{aligned}$$

The error dynamics (11)–(12) are time-varying, since the measurement q is time-varying. Now, consider the Lyapunov function candidate given by

$$V_o(t, \tilde{q}, \tilde{\nu}) = \frac{1}{2}\left(\tilde{q}^T P_o \tilde{q} + \tilde{\nu}^T M \tilde{\nu}\right) \quad (13)$$

where $P_o = P_o^T > 0$. Setting $K_{o_2}(q) = M^{-1}J(q)^T P_o$, the time derivative of V_o along the trajectories of (11)–(12) is

$$\begin{aligned} \dot{V}_o(t, \tilde{q}, \tilde{\nu}) &= -\frac{1}{2}\tilde{q}^T(P_o K_{o_1} + K_{o_1}^T P_o)\tilde{q} + \tilde{q}^T P_o J(q)\tilde{\nu} \\ &\quad -\tilde{\nu}^T M K_{o_2}(q)\tilde{q} - \frac{1}{2}\tilde{\nu}^T\left(D + D^T\right)\tilde{\nu} - \tilde{\nu}^T\left[d(\nu) - d(\hat{\nu})\right] \end{aligned}$$

Since d satisfies (5) with $P = I$, we get

$$\dot{V}_o(t, \tilde{q}, \tilde{\nu}) \leq -\frac{1}{2}\tilde{q}^T(P_o K_{o_1} + K_{o_1}^T P_o)\tilde{q} - \frac{1}{2}\tilde{\nu}^T\left(D + D^T\right)\tilde{\nu} \quad (14)$$

Clearly, for $D+D^T$ positive definite, the origin $(\tilde{q},\tilde{\nu})=(0,0)$ of (11)–(12) is UGES, provided that K_{o_1} is chosen such that $P_oK_{o_1}+K_{o_1}^TP_o$ is positive definite. Now, what happens if the damping and friction forces cannot be represented by a positive definite $D+D^T$, that is, $D+D^T$ is only positive *semi*-definite? This is, for instance, the case for offshore vessels and structures for which the hydrodynamic drag is modelled by Morison's equation, and for ships not possessing straight-line stability. In Loria and Panteley (1999), UGAS and ULES of the origin $(\tilde{q},\tilde{\nu})=(0,0)$ of (11)–(12) is shown by using Theorem 1 in Fossen et al. (2000) (also in Loria et al. (1999)), under the assumption that the trajectories of (7)–(8) are uniformly bounded. The idea of this theorem is to decompose the system into one part, for which the origin is ULES, plus a perturbation whose norm is bounded, on any compact subset of the state space, by a linear function of $\|\tilde{q}\|$. Direct application of this theorem requires a global Lipschitz condition to hold for d, but in our case, d is only locally Lipschitz. A small modification of the theorem, taking (5) into account, resolves this problem.

Proposition 1 *Let $P_o=P_o^T>0$, and suppose the observer gain matrices are chosen according to*

$$\begin{aligned} 0 &< P_oK_{o_1}+K_{o_1}^TP_o \\ K_{o_2}(q) &= M^{-1}J(q)^TP_o \end{aligned}$$

If the trajectories $\nu(t)$ are uniformly globally bounded, then the origin of (11)–(12) is uniformly globally asymptotically stable (UGAS) and uniformly locally exponentially stable (ULES). Moreover, if $D+D^T>0$ then it is uniformly globally exponentially stable (UGES).

Proof. The UGES part of the proposition is clear from equations (13) and (14). We prove the UGAS/ULES part. In reference to Definition 3, we need to prove uniform global stability (UGS) and uniform attractivity. From equations (13) and (14), and by the positive definiteness of $P_oK_{o_1}+K_{o_1}^TP_o$, there exists a strictly positive constant c_1, such that

$$\dot{V}_o(t,\tilde{q},\tilde{\nu})\leq -c_1\|\tilde{q}\|^2\leq 0 \tag{15}$$

So the origin is UGS, which means that there exists a class $\mathcal{K}_\infty$ function, γ, such that $\|x(t)\|\leq\gamma(\|x(t_0)\|)$, where we have defined $x\triangleq(\tilde{q},\tilde{\nu})$ to simplify notation. In order to prove uniform attractivity, we rewrite the system (11)–(12) as

$$\dot{x}=f(t,x)+g(t,x) \tag{16}$$

where

$$f(t,x)=\begin{bmatrix} -\tilde{q}+J(q)\tilde{\nu} \\ -M^{-1}J(q)^T\tilde{q}-M^{-1}D\tilde{\nu}-M^{-1}\left[d(\nu)-d(\hat{\nu})\right] \end{bmatrix}$$

and

$$g(t,x)=\begin{bmatrix} -K_{o_1}\tilde{q}+\tilde{q} \\ -M^{-1}J(q)^TP_o\tilde{q}+M^{-1}J(q)^T\tilde{q} \end{bmatrix}$$

We start by proving that, for any $r>0$, the origin of $\dot{x}=f(t,x)$ is ULES for initial states belonging to B_r. Define $\omega\triangleq\gamma(r)$ and consider the Lyapunov function candidate $V_2:[t_0,\infty)\times B_\omega\to\mathbb{R}_+$ defined by

$$V_2(t,x)=\frac{1}{2}\left(\tilde{q}^T\tilde{q}+\tilde{\nu}^TM\tilde{\nu}\right)-\varepsilon\tilde{q}^TJ(q)\tilde{\nu} \tag{17}$$

Since $M=M^T>0$, there exist constants c_a and c_b, with $c_a\geq c_b$, such that

$$c_b\|x\|^2-\varepsilon\tilde{q}^TJ(q)\tilde{\nu}\leq V_2(t,x)\leq c_a\|x\|^2-\varepsilon\tilde{q}^TJ(q)\tilde{\nu}$$

so, by imposing $\varepsilon \leq c_b/k_J$, we have that

$$\frac{c_b}{2}\|x\|^2 \leq V_2(t,x) \leq \frac{3c_a}{2}\|x\|^2, \qquad \forall (t,x) \in [t_0,\infty) \times B_\omega \tag{18}$$

The time derivative of $V_2(t,x)$ along trajectories of $\dot{x} = f(t,x)$ is

$$\begin{aligned}
\dot{V}_2(t,x) &\leq -\tilde{q}^T\tilde{q} + \varepsilon\tilde{q}^T J(q)M^{-1}J(q)^T\tilde{q} + \varepsilon\tilde{q}^T J(q)M^{-1}\left[d(\nu){-}d(\hat{\nu})\right] \\
&\quad +\varepsilon\tilde{q}^T J(q)M^{-1}D\tilde{\nu} - \varepsilon\tilde{q}^T\dot{J}(q,\dot{q})\tilde{\nu} + \varepsilon\tilde{\nu}^T J(q)^T\tilde{q} - \varepsilon\tilde{\nu}^T J(q)^T J(q)\tilde{\nu} \\
&\leq -\|\tilde{q}\|^2 - \varepsilon k_j^2\|\tilde{\nu}\|^2 + \varepsilon k_J^2\|M\|^{-1}\|\tilde{q}\|^2 + \varepsilon k_J\left\|M^{-1}\right\|\|d(\nu){-}d(\hat{\nu})\|\|\tilde{q}\| \\
&\quad + \left(\varepsilon k_J\left\|M^{-1}\right\|\|D\| + \varepsilon\left\|\dot{J}(q,\dot{q})\right\| + \varepsilon k_J\right)\|\tilde{q}\|\|\tilde{\nu}\|
\end{aligned}$$

Notice that since d is locally Lipschitz, we can find a Lipschitz constant, L_{d_ω}, for d on the compact set B_ω. Thus, $\|d(\nu){-}d(\hat{\nu})\| \leq L_{d_\omega}\|\tilde{\nu}\|$. Furthermore, since $\nu(t)$ is assumed to be UGB, there exists a constant, $b(\|\nu(t_0)\|)$, such that $\|\nu(t)\| \leq b(\|\nu(t_0)\|)$. Thus, using Properties 1 and 2, we get the following bound on $\left\|\dot{J}(q,\dot{q})\right\|$

$$\left\|\dot{J}(q,\dot{q})\right\| \leq L_j\|J(q)\nu\| \leq L_j k_J b(\|\nu(t_0)\|) \triangleq \beta_j$$

Defining $c = k_J\left\|M^{-1}\right\| L_{d_\omega} + k_J\left\|M^{-1}\right\|\|D\| + \beta_j + k_J$, we get

$$\dot{V}_2(t,x) \leq -\left(\frac{1}{2} - \varepsilon k_J^2\|M\|^{-1}\right)\|\tilde{q}\|^2 - \frac{1}{4}\|\tilde{q}\|^2 - \frac{\varepsilon k_j^2}{2}\|\tilde{\nu}\|^2 - \frac{1}{2}\begin{bmatrix}\|q\| \\ \|\nu\|\end{bmatrix}^T \begin{bmatrix}\frac{1}{2} & -\frac{\varepsilon c}{2} \\ -\frac{\varepsilon c}{2} & \frac{\varepsilon k_j^2}{2}\end{bmatrix}\begin{bmatrix}\|q\| \\ \|\nu\|\end{bmatrix}$$

If we pick ε such that

$$\varepsilon \leq \min\left\{\frac{c_b}{k_J}, \left(\frac{k_j}{c}\right)^2\right\}$$

then

$$\dot{V}_2(t,x) \leq -c_2\|x\|^2 \tag{19}$$

with

$$c_2 = \min\left(\frac{1}{4}, \frac{\varepsilon k_j^2}{2}\right)$$

It remains to show uniform attractivity of the full system (16). Consider the Lyapunov function candidate $\mathcal{V}: [t_0,\infty) \times B_\omega \to \mathbb{R}_+$ defined by $\mathcal{V}(t,x) \triangleq \mu V_o(t,x) + V_2(t,x)$ where $\mu > 0$ is to be specified later. From equations (13) and (18), it is clear that there exist constants c_a' and c_b', with $c_a' > c_b'$, such that

$$c_b'\|x\|^2 \leq \mathcal{V}(t,x) \leq c_a'\|x\|^2 \tag{20}$$

From equations (15), (16), and (19) we have

$$\dot{\mathcal{V}}(t,x) \leq -c_1\mu\|\tilde{q}\|^2 - c_2\|x\|^2 + \left\|\frac{\partial V_2}{\partial x}(t,x)\right\|\|g(t,x)\| \tag{21}$$

Using Property 1 we can find a positive constant c_3, such that

$$\|g(q,\nu,t)\| \leq c_3\|\tilde{q}\| \tag{22}$$

Let $j_i(q)$ denote the i^{th} column of $J(q)$, that is

$$J(q) = \left[\begin{array}{cccc} j_1(q) & j_2(q) & \cdots & j_n(q) \end{array} \right]$$

The Jacobian of $j_i(q)$, $\frac{\partial j_i(q)}{\partial q}$, is bounded uniformly in q, otherwise Property 2 would not hold. Thus, we can find a constant, $\beta_{\partial j}$, such that

$$\left\| \frac{\partial j_i(q)}{\partial q} \right\| \leq \beta_{\partial j}, \ \ \forall q \in \mathbb{R}^n$$

The Jacobian of $V_2(t,x)$ is

$$\begin{aligned} \frac{\partial V_2}{\partial x}(t,x) &= \left[\begin{array}{cc} \tilde{q}^T - \varepsilon \tilde{\nu}^T \frac{\partial}{\partial \tilde{q}}(J(\tilde{q}+\hat{q})^T \tilde{q}) & \tilde{\nu}^T M - \varepsilon \tilde{q}^T J(q) \end{array} \right] \\ &= \left[\begin{array}{cc} \tilde{q}^T - \varepsilon \tilde{\nu}^T \left(J(q)^T + \left[\begin{array}{c} \tilde{q}^T \frac{\partial j_1(\tilde{q}+\hat{q})}{\partial \tilde{q}} \\ \tilde{q}^T \frac{\partial j_2(\tilde{q}+\hat{q})}{\partial \tilde{q}} \\ \vdots \\ \tilde{q}^T \frac{\partial j_n(\tilde{q}+\hat{q})}{\partial \tilde{q}} \end{array} \right] \right) & \tilde{\nu}^T M - \varepsilon \tilde{q}^T J(q) \end{array} \right] \end{aligned}$$

By noticing that

$$\left\| \left[\begin{array}{c} \tilde{q}^T \frac{\partial j_1(\tilde{q}+\hat{q})}{\partial \tilde{q}} \\ \tilde{q}^T \frac{\partial j_2(\tilde{q}+\hat{q})}{\partial \tilde{q}} \\ \vdots \\ \tilde{q}^T \frac{\partial j_n(\tilde{q}+\hat{q})}{\partial \tilde{q}} \end{array} \right] \right\| \leq \sqrt{n} \beta_{\partial j} \left\| \tilde{q} \right\| \leq \sqrt{n} \beta_{\partial j} \omega$$

and, by using Property 1, we get

$$\begin{aligned} \left\| \frac{\partial V_2}{\partial x}(x,t) \right\|^2 \leq \; & \left(1 + \varepsilon^2 k_J^2\right) \left\|\tilde{q}\right\|^2 + 2\varepsilon \left(k_J \left\|M\right\| + \left(k_J + \sqrt{n}\beta_{\partial j}\omega\right)\right) \left\|\tilde{q}\right\| \left\|\tilde{\nu}\right\| \\ & + \left(\left\|M\right\| + \varepsilon^2 \left(k_J + \sqrt{n}\beta_{\partial j}\omega\right)^2 \right) \left\|\tilde{\nu}\right\|^2 \end{aligned}$$

Thus, we can find a positive constant c_4, such that

$$\left\| \frac{\partial V_2}{\partial x}(x,t) \right\| \leq c_4 \left\|x\right\| \tag{23}$$

Inserting the bounds (22) and (23), into (21), we get

$$\begin{aligned} \dot{\mathcal{V}}(t,x) &\leq -c_1 \mu \left\|\tilde{q}\right\|^2 - c_2 \left\|x\right\|^2 + c_3 c_4 \left\|x\right\| \left\|\tilde{q}\right\| \\ &= -c_1 \mu \left\|\tilde{q}\right\|^2 - \frac{c_2}{2} \left\|x\right\|^2 + \frac{(c_3 c_4)^2}{2c_2} \left\|\tilde{q}\right\|^2 - \left(\sqrt{\frac{c_2}{2}} \left\|x\right\| - \frac{c_3 c_4}{\sqrt{2c_2}} \left\|\tilde{q}\right\|\right)^2 \\ &\leq -\left(c_1 \mu - \frac{(c_3 c_4)^2}{2c_2}\right) \left\|\tilde{q}\right\|^2 - \frac{c_2}{2} \left\|x\right\|^2 \end{aligned} \tag{24}$$

Thus, for sufficiently large μ, $\dot{\mathcal{V}}(t,x)$ is negative definite on $[t_0,\infty)\times B_\omega$. From equation (20), and the last term in equation (24), we conclude that the origin $x=0$ of (16) is ULES, that is, there exist strictly positive constants k and γ_3 such that

$$\|x(t)\| \leq k\,\|x(t_0)\|\, e^{-\gamma_3(t-t_0)}$$

for all $t \geq t_0$ and $x(t_0)\in B_r$. Now, given $\epsilon > 0$ (recall that $r>0$ is already given), we have that (3) holds with $T = \ln\left(\frac{\epsilon}{kr}\right)/\gamma_3$. ■

Under an additional assumption on $J(q)$, and by following the lines of Arcak and Kokotović (1999a), we can find an observer that renders the error dynamics UGES, even in the case of only positive semi-definite $D+D^T$.

Assumption 1 *A locally Lipschitz function, $\phi:\mathbb{R}^n\to\mathbb{R}^n$, is known, such that*

$$\frac{\partial\phi}{\partial q}(q)\,J(q) + \left(\frac{\partial\phi}{\partial q}(q)\,J(q)\right)^T > \epsilon I,\quad \forall q\in\mathbb{R}^n$$

for some $\epsilon > 0$.

We will use the knowledge of such a function to perform a change of variables. Define the new variable as follows

$$y = \nu - \phi(q)$$

where ϕ satisfies Assumption 1. Writing the system (7)–(8) in terms of q and y, we obtain

$$\begin{aligned}\dot{q} &= J(q)y + J(q)\phi(q)\\ \dot{y} &= -M^{-1}\left(D(y+\phi(q)) + d(y+\phi(q)) + v(q) - \tau\right) - \frac{\partial\phi}{\partial q}\left(J(q)y + J(q)\phi(q)\right)\end{aligned}$$

Now, consider the observer

$$\begin{aligned}\dot{\hat{q}} &= J(q)\hat{y} + J(q)\phi(q) + K_{o_1}\tilde{q} &(25)\\ \dot{\hat{y}} &= -M^{-1}\left(D(\hat{y}+\phi(q)) + d(\hat{y}+\phi(q)) + v(q) - \tau\right)\\ &\quad -\frac{\partial\phi}{\partial q}\left(J(q)\hat{y} + J(q)\phi(q)\right) + K_{o_2}(q)\tilde{q} &(26)\end{aligned}$$

with error dynamics, in terms of $\tilde{q} \triangleq q-\hat{q}$ and $\tilde{y}\triangleq y-\hat{y}$, as follows

$$\begin{aligned}\dot{\tilde{q}} &= -K_{o_1}\tilde{q} + J(q)\tilde{y} &(27)\\ \dot{\tilde{y}} &= -K_{o_2}(q)\tilde{q} - M^{-1}D\tilde{y} - M^{-1}\left[d(y+\phi(q)) - d(\hat{y}+\phi(q))\right] - \frac{\partial\phi}{\partial q}J(q)\tilde{y} &(28)\end{aligned}$$

Proposition 2 *Suppose the observer gain matrices in (25)–(26) are chosen as in Proposition 1. If Assumption 1 holds, then the origin of (27)–(28) is uniformly globally exponentially stable (UGES).*

Proof. Using the Lyapunov function candidate $V_o(t,\tilde{q},\tilde{y}) = \frac{1}{2}\left(\tilde{q}^TP_o\tilde{q} + \tilde{y}^TM\tilde{y}\right)$, and setting $K_{o_2}(q) = M^{-1}J(q)^TP_o$, the time derivative of V_o along the trajectories of (27)–(28) is

$$\begin{aligned}\dot{V}_o(t,\tilde{q},\tilde{y}) &= -\frac{1}{2}\tilde{q}^T(P_oK_{o_1} + K_{o_1}^TP_o)\tilde{q} + \tilde{q}^TP_oJ(q)\tilde{y} - \tilde{y}^TJ(q)^TP_o\tilde{q} - \frac{1}{2}\tilde{y}^T\left(D+D^T\right)\tilde{y}\\ &\quad -\tilde{y}^T\left[d(y+\phi(q)) - d(\hat{y}+\phi(q))\right] - \frac{1}{2}\tilde{y}^T\left(\frac{\partial\phi}{\partial q}J(q) + \left(\frac{\partial\phi}{\partial q}J(q)\right)^T\right)\tilde{y}\end{aligned}$$

Since d satisfies (5) with $P = I$, we get

$$\dot{V}_o(t, \tilde{q}, \tilde{\nu}) \leq -\frac{1}{2}\tilde{q}^T(P_oK_{o_1} + K_{o_1}^T P_o)\tilde{q} - \frac{1}{2}\tilde{y}^T \left(\frac{\partial \phi}{\partial q}J(q) + \left(\frac{\partial \phi}{\partial q}J(q)\right)^T\right)\tilde{y}$$

In view of Assumption 1, UGES of the origin $(\tilde{q}, \tilde{y}) = 0$ of (27)–(28) follows from standard results (Khalil 1996, Corollary 3.4). ■

4.2 Controller design

Now, moving on to controller design, we define the virtual tracking error as

$$\hat{z} \triangleq J(q)\hat{\nu} - \dot{q}_d + K_p q_e$$

where $q_e \triangleq q - q_d$ and $K_p + K_p^T > 0$. Then,

$$\dot{q}_e = -K_p q_e + \hat{z} + J(q)\tilde{\nu} \tag{29}$$

and

$$\dot{\hat{z}} = \dot{J}(q, \dot{q})\hat{\nu} + J(q)M^{-1}\left(\tau + MK_{o_2}(q)\tilde{q} - D\hat{\nu} - d(\hat{\nu}) - v(q)\right) - \ddot{q}_d + K_p\left(J(q)\nu - \dot{q}_d\right) \tag{30}$$

The theorem below covers the general case, that is, when $D + D^T$ is only positive semi-definite, and we cannot find a ϕ that satisfies Assumption 1. In this case, we achieve UGAS of the origin of the error dynamics.

Theorem 1 *Let $P_o = P_o^T > 0$, and set $K_{o_2} = M^{-1}J(q)^T P_o$. Suppose that the matrices K_{o_1}, K_p, and K_d, are chosen such that*

$$\begin{aligned} P_oK_{o_1} + K_{o_1}^T P_o &> 0 \\ K_p &= K_p^T > 0 \\ K_d &= K_d^T > 0 \end{aligned}$$

Then, the nonlinear output feedback tracking controller

$$\begin{aligned} \tau &= MJ(q)^{-1}\left\{-q_e - \left(K_d + \|q_e\|^2 I + \|\hat{z}\|^2 I\right)\hat{z} + \ddot{q}_d + K_p\dot{q}_d - K_pJ(q)\hat{\nu} - \dot{J}(q, J(q)\hat{\nu})\hat{\nu}\right\} \\ &\quad + v(q) + D\hat{\nu} + d(\hat{\nu}) \end{aligned} \tag{31}$$

renders the origin $\xi = (\tilde{q}, \tilde{\nu}, q_e, \hat{z}) = 0$ of the error dynamics (11)–(12) and (29)–(30) uniformly globally asymptotically stable (UGAS).

Proof. Inserting the control law (31) into (30), we obtain

$$\begin{aligned} \dot{\hat{z}} &= -q_e - \left(K_d + \|q_e\|^2 I + \|\hat{z}\|^2 I\right)\hat{z} \\ &\quad + J(q)M^{-1}J(q)^T P_0\tilde{q} + \left[\dot{J}(q, J(q)\nu) - \dot{J}(q, J(q)\hat{\nu})\right]\hat{\nu} + K_pJ(q)\tilde{\nu} \end{aligned} \tag{32}$$

The derivative of $V_c(t, q_e, \hat{z}) = \frac{1}{2}\|q_e\|^2 + \frac{1}{2}\|\hat{z}\|^2$ along the trajectories of (29) and (32) is

$$\begin{aligned} \dot{V}_c(t, q_e, \hat{z}) &= -q_e^T K_p q_e - \hat{z}^T K_d\hat{z} - \|q_e\|^2\|\hat{z}\|^2 - \|\hat{z}\|^4 + \hat{z}^T J(q)M^{-1}J(q)^T P_0\tilde{q} \\ &\quad + \hat{z}^T\left[\dot{J}(q, J(q)\nu) - \dot{J}(q, J(q)\hat{\nu})\right]\hat{\nu} + \hat{z}^T K_pJ(q)\tilde{\nu} + q_e^T J(q)\tilde{\nu} \end{aligned} \tag{33}$$

Using Properties 1 and 2, we get

$$\hat{z}^T \left[\dot{J}(q, J(q)\nu) - \dot{J}(q, J(q)\hat{\nu})\right] \hat{\nu} \leq L_j k_J \|\hat{z}\| \|\tilde{\nu}\| \|\hat{\nu}\| \tag{34}$$

and, since $J(q)\hat{\nu} = \hat{z} + \dot{q}_d - K_p q_e$, we get

$$\|\hat{\nu}\| \leq k_{J^{-1}} \|\hat{z}\| + k_{J^{-1}} \|K_p\| \|q_e\| + k_{J^{-1}} \beta_d \tag{35}$$

where $k_{J^{-1}}$ is an upper bound on $\left\|J(q)^{-1}\right\|$ (which exists by Property 1). Inequalities (34) and (35) imply

$$\hat{z}^T \left[\dot{J}(q, J(q)\nu) - \dot{J}(q, J(q)\hat{\nu})\right] \hat{\nu} \leq L_j k_J k_{J^{-1}} \left(\|\hat{z}\| + \|K_p\| \|q_e\| + \beta_d\right) \|\hat{z}\| \|\tilde{\nu}\| \tag{36}$$

Substitution of inequality (36), along with the inequalities

$$\hat{z}^T J(q) M^{-1} J(q)^T P_0 \tilde{q} \quad \leq \quad k_J^2 \left\|M^{-1}\right\| \|P_0\| \|\hat{z}\| \|\tilde{q}\| \tag{37}$$

$$\hat{z}^T K_p J(q) \tilde{\nu} \quad \leq \quad k_J \|K_p\| \|\hat{z}\| \|\tilde{\nu}\| \tag{38}$$

$$q_e^T J(q) \tilde{\nu} \quad \leq \quad k_J \|q_e\| \|\tilde{\nu}\| \tag{39}$$

into equation (33), yields

$$\begin{aligned}
\dot{V}_c \quad \leq \quad & -\frac{1}{2} q_e^T K_p q_e - \frac{1}{2} \hat{z}^T K_d \hat{z} + \left(-\frac{1}{4} \lambda_{\min}(K_d) \|\hat{z}\|^2 + k_J^2 \left\|M^{-1}\right\| \|P_0\| \|\hat{z}\| \|\tilde{q}\|\right) \\
& + \left(-\|\hat{z}\|^4 + L_j k_J k_{J^{-1}} \|\hat{z}\|^2 \|\tilde{\nu}\|\right) + \left(-\|q_e\|^2 \|\hat{z}\|^2 + L_j k_J k_{J^{-1}} \|K_p\| \|q_e\| \|\hat{z}\| \|\tilde{\nu}\|\right) \\
& + \left(-\frac{1}{2} \lambda_{\min}(K_p) \|q_e\|^2 + k_J \|q_e\| \|\tilde{\nu}\|\right) \\
& + \left(-\frac{1}{4} \lambda_{\min}(K_d) \|\hat{z}\|^2 + \left(L_j k_J k_{J^{-1}} \beta_d + k_J \|K_p\|\right) \|\hat{z}\| \|\tilde{\nu}\|\right)
\end{aligned}$$

where $\lambda_{\min}(K_?)$ denotes the smallest eigenvalue of the matrix $K_?$. Upon completion of squares, one can find a constant c such that

$$\dot{V}_c(t, q_e, \hat{z}) \leq -\frac{1}{2} q_e^T K_p q_e - \frac{1}{2} \hat{z}^T K_d \hat{z} + c \left\|(\tilde{q}, \tilde{\nu})\right\|^2 \tag{40}$$

which proves, using standard results (Khalil 1996, Theorem 5.2), that the $(q_e, \hat{z})$-subsystem is ISS with input $(\tilde{q}, \tilde{\nu})$. From the proof of Proposition 1, $\|(\tilde{q}, \tilde{\nu})\|$ is UGB, so it follows from (40) that q_e and $\hat{z}$ are UGB. Therefore, ν is UGB. The theorem now follows from (Khalil 1996, Lemma 5.6) along with Proposition 1. ∎

In the control law (31) of Theorem 1, the variables q_e and $\hat{z}$ are raised to the third power in order to dominate the estimation error. This may be undesirable due to practical issues such as measurement noise, saturation in the actuators, and unmodeled actuator dynamics. The next result covers the case when an exponentially convergent observer is available, in which case we can achieve UGAS of the origin of the overall system with less control effort. Moreover, we achieve exponential convergence to any ϵ-neighborhood of the origin. Before we state the theorem, we need the following lemma.

Lemma 2 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0 \tag{41}$$

where $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x. Let $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable function such that

$$k_1 \|x\|^c \leq V(t,x) \leq k_2 \|x\|^c \tag{42}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3 \|x\|^c + g(\|x\|)\, \sigma(\|x(t_0)\|, t - t_0) \tag{43}$$

$\forall t \geq t_0$, $\forall x \in \mathbb{R}^n$, where k_1, k_2, k_3, and c, are strictly positive constants, $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous, and σ is a class $\mathcal{KL}$ function satisfying

$$\int_{t_0}^{\infty} \sigma(r,s)\, ds \leq \sigma_\infty r \tag{44}$$

for some constant σ_∞. Suppose that there exist constants $k > 0$, and $r \geq 0$ such that $k\|x\|^c \geq g(\|x\|)$, $\forall \|x\| \geq r$. Then, the equilibrium point $x = 0$ of (41) is uniformly globally asymptotically stable (UGAS).

Proof. First we show that solutions can be continued for all time. From (42) and (43), we get

$$\begin{aligned} \|x(t)\|^c &\leq \frac{V(t,x)}{k_1} \leq \frac{V(t_0,x_0)}{k_1} + \frac{1}{k_1}\int_{t_0}^{t} \dot{V}(\tau,x) d\tau \\ &\leq \frac{k_2}{k_1}\|x_0\|^c + \frac{1}{k_1}\int_{t_0}^{t} g(\|x\|)\, \sigma(\|x_0\|, t - t_0)\, d\tau \end{aligned} \tag{45}$$

Define

$$\beta_r = \sup_{\|x\| < r} g(\|x\|)$$

By continuity of g on $\mathbb{R}_+$, β_r is finite. We then get

$$\|x(t)\|^c \leq \frac{k_2}{k_1}\|x_0\|^c + \frac{\beta_r \sigma_0}{k_1}(t - t_0) + \int_{t_0}^{t} \frac{k\sigma_0}{k_1}\|x(\tau)\|^c d\tau$$

where we have set $\sigma_0 = \sigma(\|x_0\|, 0)$, and used the fact that $k\|x\|^c \geq g(\|x\|)$ for sufficiently large $\|x\|$. Using Gronwall's inequality (Khalil 1996, Lemma 2.1), it follows that

$$\|x(t)\|^c \leq \frac{k_2}{k_1}\|x_0\|^c + \frac{\beta_r \sigma_0}{k_1}(t - t_0) + \frac{k\sigma_0}{k_1}\left(\frac{k_2}{k_1}\|x_0\|^c (t - t_0) + \frac{\beta_r \sigma_0}{2k_1}(t - t_0)^2\right) e^{\frac{k\sigma_0}{k_1}(t - t_0)}$$

which shows that solutions exist for all $t \geq t_0$. We proceed to show that the solutions are in fact bounded. Since $\sigma \in \mathcal{KL}$ there exists a time $t_* \geq t_0$, dependent on the initial state, such that $\sigma(\|x_0\|, t_* - t_0) < \frac{k_3}{2k}$. Thus, we get

$$\dot{V} \leq -k_3\|x\|^c + \frac{k_3 g(\|x\|)}{2k} \leq -\frac{k_3}{2}\|x\|^c, \ \forall \|x\| \geq r, \ \forall t > t_*$$

Again we have used the fact that $k\|x\|^c \geq g(\|x\|)$ for sufficiently large $\|x\|$. This shows that the set $\mathcal{S} = \{x : \|x\| \leq r\}$ is globally attractive, that is, the distance between x and $\mathcal{S}$ tends to 0 as $t \to \infty$, for all initial states x_0. Thus, boundedness of solutions follows. We finish the proof by showing uniform attractivity. Since the solutions are bounded, given β, there exists $\beta_b > 0$, such that for all $\|x_0\| < \beta$, $\|x(t)\| < \beta_b$ for all $t > t_0$. Now, define

$$\beta_g = \sup_{\|x\|<\beta_b} g(\|x\|)$$

By continuity of g on $\mathbb{R}_+$, β_g is finite. From (42)–(43), and (45), we get

$$\begin{aligned}\|x(t)\|^c &\leq \frac{k_2}{k_1}\|x_0\|^c + \int_{t_0}^{t}\left[-\frac{k_3}{k_1}\|x(\tau)\|^c + \frac{1}{k_1}g(\|x(\tau)\|)\,\sigma(\|x_0\|, \tau - t_0)\right]d\tau \\ &\leq \frac{k_2}{k_1}\|x_0\|^c + \int_{t_0}^{t}\left[-\frac{k_3}{k_1}\|x(\tau)\|^c + \frac{\beta_g}{k_1}\sigma(\|x_0\|, \tau - t_0)\right]d\tau \\ &\leq \frac{k_2}{k_1}\beta^c + \frac{\beta_g\sigma_\infty\beta}{k_1} + \int_{t_0}^{t}\left[-\frac{k_3}{k_1}\|x(\tau)\|^c\right]d\tau\end{aligned}$$

where we have used (44) in the last step. Using Gronwall's inequality, it follows that

$$\|x(t)\|^c \leq \left(\frac{k_2}{k_1}\beta^c + \frac{\beta_g\sigma_\infty\beta}{k_1}\right)e^{-\frac{k_3}{k_1}(t-t_0)}$$

and finally that

$$\|x(t)\| \leq \left(\frac{k_2}{k_1}\beta^c + \frac{\beta_g\sigma_\infty\beta}{k_1}\right)^{\frac{1}{c}}e^{-\frac{k_3}{ck_1}(t-t_0)} \tag{46}$$

■

Theorem 2 *Let $P_o = P_o^T > 0$ be a symmetric, positive definite matrix, and set $K_{o_2} = M^{-1}J(q)^T P_o$. Suppose that the matrices K_{o_1}, K_p, and K_d, are chosen such that*

$$\begin{aligned}P_oK_{o_1} + K_{o_1}^T P_o &> 0 \\ K_p &= K_p^T > 0 \\ K_d &= K_d^T > 0\end{aligned}$$

If Assumption 1 holds (respectively, $D + D^T$ is positive definite), then, the nonlinear output feedback tracking controller

$$\tau = MJ(q)^{-1}\left\{-q_e - K_d\hat{z} + \ddot{q}_d + K_p\dot{q}_d - K_pJ(q)\hat{\nu} - \dot{J}(q, J(q)\hat{\nu})\hat{\nu}\right\} + v(q) + D\hat{\nu} + d(\hat{\nu}) \tag{47}$$

renders the origin $\xi = (\tilde{q}, \tilde{y}, q_e, \hat{z}) = 0$ (respectively, $\xi = (\tilde{q}, \tilde{\nu}, q_e, \hat{z}) = 0$) of the error dynamics (27)–(28) (respectively, (11)–(12)) and (29)–(30) uniformly globally asymptotically stable (UGAS).

Proof. Inserting the control law (47) into (30), we obtain

$$\dot{\hat{z}} = -q_e - K_d\hat{z} + J(q)M^{-1}J(q)^T P_0\tilde{q} + \left[\dot{J}(q, J(q)\nu) - \dot{J}(q, J(q)\hat{\nu})\right]\hat{\nu} + K_pJ(q)\tilde{\nu} \tag{48}$$

The derivative of $V_c(t, q_e, \hat{z}) = \frac{1}{2}\|q_e\|^2 + \frac{1}{2}\|\hat{z}\|^2$ along the trajectories of (29) and (48) is

$$\begin{aligned} \dot{V}_c(t, q_e, \hat{z}) &= -q_e^T K_p q_e - \hat{z}^T K_d \hat{z} + \hat{z}^T J(q) M^{-1} J(q)^T P_0 \tilde{q} \\ &\quad + \hat{z}^T \left[\dot{J}(q, J(q)\nu) - \dot{J}(q, J(q)\hat{\nu}) \right] \hat{\nu} + \hat{z}^T K_p J(q) \tilde{\nu} + q_e^T J(q) \tilde{\nu} \end{aligned} \tag{49}$$

As in the proof of Theorem 1, substitution of inequalitities (36), and (37)-(39) into equation (49), and noticing that $\|\hat{z}\| \leq \|\xi\|$ and $\|q_e\| \leq \|\xi\|$, yields

$$\dot{V}_c \leq -q_e^T K_p q_e - \hat{z}^T K_d \hat{z} + \left(c_1 \|\xi\| + c_2 \|\xi\|^2 \right) \|(\tilde{q}, \tilde{\nu})\|$$

where we have defined

$$\begin{aligned} c_1 &\triangleq k_J \left(k_J \left\| M^{-1} \right\| \|P_0\| + L_{\dot{J}} k_{J^{-1}} \beta_d + \|K_p\| + 1 \right) \\ c_2 &\triangleq L_{\dot{J}} k_J k_{J^{-1}} \left(1 + \|K_p\| \right) \end{aligned}$$

In view of Proposition 2 (respectively, the UGES part of Proposition 1), the positive definiteness of K_p and K_d, ensures the existence of a strictly positive constant c, such that the time derivative of $V = V_o + V_c$ is bounded above as follows

$$\dot{V} \leq -c\|\xi\|^2 + \left(c_1 \|\xi\| + c_2 \|\xi\|^2 \right) \|(\tilde{q}, \tilde{\nu})\|$$

Proposition 2 (respectively, the UGES part of Proposition 1) provides the following bound on $\|(\tilde{q}, \tilde{\nu})\|$

$$\|(\tilde{q}, \tilde{\nu})\| \leq k_1 \|(\tilde{q}(t_0), \tilde{\nu}(t_0))\| e^{-\gamma(t-t_0)} \leq k_1 \|\xi(t_0)\| e^{-\gamma(t-t_0)}$$

where k_1 and γ are strictly positive constants. Thus, we get

$$\dot{V} \leq -c\|\xi\|^2 + g(\|\xi\|)\sigma(\|\xi(t_0)\|, t - t_0)$$

where we have defined

$$g(\rho) \triangleq c_1 \rho + c_2 \rho^2$$

and

$$\sigma(\rho, \tau) \triangleq k_1 \rho e^{-\gamma \tau}$$

Clearly, σ is integrable in its second argument on $\mathbb{R}_+$. Also, if we pick $r = 1$ and $k_2 = c_1 + c_2$, we have $g(\rho) \leq k_2 \rho^2$, for all $\rho \geq r$. Thus, we can apply Lemma 2 to conclude that the origin $\xi = 0$ is UGAS. ■

Remark 1 *Notice from (46), that for any $\epsilon > 0$, we can find a constant k_ϵ, such that*

$$\|x(t)\| \leq k_\epsilon \|x(t_0)\| e^{-\frac{k_3}{c k_1}(t-t_0)}, \quad \forall \|x(t)\| > \epsilon$$

In other words, we have exponential convergence of the trajectories of (27)–(28) (respectively, (11)–(12)) and (29)–(30) to any ϵ-neighborhood of the origin.

Remark 2 *Notice that in both theorems above, a separation principle holds in the sense that the proposed controller, which clearly stabilizes the system when the full state is available, in conjunction with the proposed observer, renders the origin of the overall system UGAS. In addition, the controller gains and the observer gains can be chosen independently of each other.*

5 Examples

Example 1 *Consider the mass-spring-damper system*

$$m\ddot{x} + d_1\dot{x} + d_2\left|\dot{x}\right|\dot{x} + d_3\left|\dot{x}\right|^2\dot{x} + d_4\left|\dot{x}\right|^3\dot{x} + \cdots + d_n\left|\dot{x}\right|^{n-1}\dot{x} + k(x)x = u \tag{50}$$

where $m > 0$, d_1, d_2, d_3,...,$d_n \geq 0$, $n \in \mathbb{N}$, $k(x)$ *is a spring constant (possibly nonlinear), and* u *is the control input. Defining* $\nu = \dot{x}$, *and* $q = x$, *(50) can be written in the form (7)–(8) with*

$$J(q) = 1, \quad M = m, \quad D = d_1,$$
$$d(\nu) = d_2\left|\nu\right|\nu + d_3\left|\nu\right|^2\nu + d_4\left|\nu\right|^3\nu + \cdots + d_n\left|\nu\right|^{n-1}\nu, \quad v(q) = k(q)q, \quad \text{and } \tau = u$$

It is clear that $J(q)$ *satisfies Properties 1 and 2, and since* $d/d\nu(\left|\nu\right|^i\nu) = (i+1)\left|\nu\right|^i \geq 0$ *for all* $\nu \in \mathbb{R}$ *and for all* $i \in 1, 2, \ldots, n$, $d(\nu)$ *satisfies (5) by Lemma 1. Thus, Theorem 1 is applicable. Clearly, if* $d_1 > 0$ *then Theorem 2 is also applicable. In fact, Theorem 2, is applicable for* $d_1 = 0$ *as well, since Assumption 1 holds (pick* $\phi(q) = q$*).*

Example 2 (omni-directional intelligent navigator (ODIN)) *ODIN is a spherical autonomous underwater vehicle developed at the Autonomous Systems Laboratory at the University of Hawaii (Choi et al. 1995). Motion on a horizontal plane (constant depth) is governed by (Kardash 1998)*

$$\begin{aligned} M\dot{\xi} + C(\xi)\xi + D(\xi)\xi &= \tau \\ \dot{\eta} &= J(\eta)\xi \end{aligned}$$

where

$$\begin{aligned} M &= \begin{bmatrix} 2m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & \frac{8}{15}\pi\rho R^5 \end{bmatrix}, \quad C(\xi) = \begin{bmatrix} 0 & 0 & -2mv \\ 0 & 0 & 2mu \\ 2mv & -2mu & 0 \end{bmatrix} \\ D(\xi) &= \begin{bmatrix} d_t\left|(u,v)\right| & 0 & 0 \\ 0 & d_t\left|(u,v)\right| & 0 \\ 0 & 0 & d_1\left|r\right| + d_2 \end{bmatrix}, \quad J(\eta) = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$\xi = [\ u \quad v \quad r\ ]^T$ *is the velocity in body-fixed coordinates,* $\eta = [\ x \quad y \quad \psi\ ]^T$ *is the position and orientation in earth-fixed coordinates, and* τ *is the control input.* $s\psi$ *and* $c\psi$ *denote the sine of* ψ *and cosine of* ψ*, respectively.* M *is the mass matrix including hydrodynamic added mass,* C *is the Coriolis and centripetal matrix including hydrodynamic added mass, and* D *is the hydrodynamic damping matrix.* m *and* R *are the mass and radius of the ODIN, respectively.* d_t, d_1 *and* d_2 *are positive constants for the hydrodynamic damping forces, and* ρ *is the water density. This system fits directly into the form of (7)–(8) by setting* $q = \eta$, $\nu = \xi$, $v(q) = 0$, *and* $d(\nu) = (C(\nu) + D(\nu))\nu$. $J(\eta)$ *is orthogonal, and clearly satisfies Properties 1 and 2. However,* $d(\xi)$ *does not satisfy the condition of Lemma 1, and in particular, it can be shown that (5) does not hold. This problem is resolved by transforming the equation of motion into earth-fixed coordinates as follows*

$$\begin{aligned} M_\eta &= J(\psi)MJ^T(\psi) = M \\ C_\eta(\dot{\eta}) &= J(\psi)\left[C(\xi) - MJ^T(\psi)\dot{J}(\psi)\right]J^T(\psi) = \begin{bmatrix} 0 & 2m\dot{\psi} & -2m\dot{y} \\ -2m\dot{\psi} & 0 & 2m\dot{x} \\ 2m\dot{y} & -2m\dot{x} & 0 \end{bmatrix} \end{aligned}$$

$$D_\eta(\eta,\dot\eta) = J(\psi)\left[D(\xi)\right]J^T(\psi)$$
$$= \begin{bmatrix} c^2\psi d_t|u| + s^2\psi d_t|v| & c\psi s\psi d_t|u| - s\psi c\psi d_t|v| & 0 \\ s\psi c\psi d_t|u| - s\psi c\psi d_t|v| & s^2\psi d_t|u| + c^2\psi d_t|v| & 0 \\ 0 & 0 & d_1\left|\dot\psi\right| + d_2 \end{bmatrix}$$

where we have kept $u = \dot x c\psi + \dot y s\psi$ and $v = -\dot x s\psi + \dot y c\psi$ in the expression for D_η for notational simplicity. Defining $q = \eta$, and $\nu = \dot\eta$, the elements of (7)–(8) become

$$\begin{aligned} J(q) &= I, \quad M = \begin{bmatrix} 2m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & \frac{8}{15}\pi\rho R^5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_2 \end{bmatrix} \\ d(q,\nu) &= \left[C_\eta(\nu) + D_\eta(q,\nu)\right]\nu = \begin{bmatrix} c\psi d_t|u|u - s\psi d_t|v|v \\ s\psi d_t|u|u + c\psi d_t|v|v \\ d_1\left|\dot\psi\right|\dot\psi \end{bmatrix}, \quad v(q) = 0 \end{aligned}$$

The fact that q appears in the expression for d, does not present a problem, since q is measured. In order to apply Lemma 1, we need the Jacobian of d (with respect to ν) which is

$$\frac{\partial d}{\partial \nu} = \begin{bmatrix} 2c^2\psi d_t|u| + 2s^2\psi d_t|v| & 2s\psi c\psi d_t|u| - 2s\psi c\psi d_t|v| & 0 \\ 2s\psi c\psi d_t|u| - 2s\psi c\psi d_t|v| & 2s^2\psi d_t|u| + 2c^2\psi d_t|v| & 0 \\ 0 & 0 & 2d_1\left|\dot\psi\right| \end{bmatrix}$$

so we get

$$\left[\frac{\partial d}{\partial \nu}\right] + \left[\frac{\partial d}{\partial \nu}\right]^T = \begin{bmatrix} 4c^2\psi d_t|u| + 4s^2\psi d_t|v| & 4s\psi c\psi d_t|u| - 4s\psi c\psi d_t|v| & 0 \\ 4s\psi c\psi d_t|u| - 4s\psi c\psi d_t|v| & 4s^2\psi d_t|u| + 4c^2\psi d_t|v| & 0 \\ 0 & 0 & 4d_1\left|\dot\psi\right| \end{bmatrix} \tag{51}$$

Since all the principal submatrices of (51) have nonnegative determinants, (51) is positive semi-definite for all $(\dot x, \dot y, \dot\psi) \in \mathbb{R}^3$ (and for all ψ). Therefore, by Lemma 1, (5) holds for d with $P = I$. We can now apply Theorem 1. D is not positive definite, but Assumption 1 holds (simply choose ϕ equal to the identity on $\mathbb{R}^3$). Thus, we can also apply Theorem 2. Based on Theorem 2, we construct the following controller

$$\begin{aligned} \begin{bmatrix} \dot{\hat q} \\ \dot{\hat y} \end{bmatrix} &= \begin{bmatrix} -K_{o_1} & I \\ -M^{-1}P_o & -(I + K_p + K_d) \end{bmatrix} \begin{bmatrix} \hat q \\ \hat y \end{bmatrix} \\ &\quad + \begin{bmatrix} K_{o_1} + I & 0 & 0 & 0 \\ M^{-1}P_o - 2I - K_p - K_d - K_dK_p & I + K_dK_p & K_p + K_d & I \end{bmatrix} \begin{bmatrix} q \\ q_d \\ \dot q_d \\ \ddot q_d \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \tau &= d(\hat y + q) + \left(D - M\left(K_d + K_p\right)\right)\hat y \\ &\quad + \begin{bmatrix} D - M\left(I + K_p + K_d + K_dK_p\right) & M\left(I + K_dK_p\right) & M\left(K_p + K_d\right) & M \end{bmatrix} \begin{bmatrix} q \\ q_d \\ \dot q_d \\ \ddot q_d \end{bmatrix} \end{aligned}$$

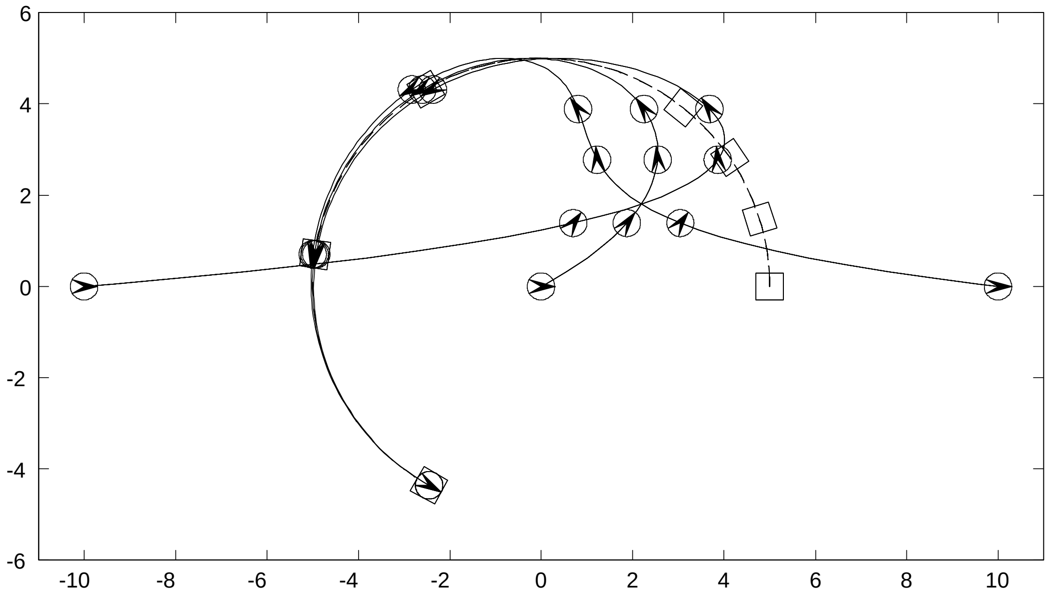

Figure 1: Simulation of output tracking of the ODIN. The desired trajectory (dashed line) is constant velocity of $1.5m/s$ on a circle with radius $5m$, starting in the location $(x, y) = (5, 0)$ at $t = 0$. The desired heading is in the direction of travel. Trajectories of the ODIN are shown for three different initial conditions: 1) $(x, y, \psi) = (-10, 0, 0)$; 2) $(x, y, \psi) = (0, 0, 0)$; and 3) $(x, y, \psi) = (10, 0, 0)$. The initial velocity is zero in all cases. The initial condition of the observer is zero position and zero velocity. A box is plotted at the desired position for seven time instances. At the same time instances, circles with arrows are plotted indicating the position and heading of the ODIN.

Simulations (see figure 1) confirm that tracking is obtained for this test case. The numerical values used in the simulations are summarized below (the physical parameters for the ODIN model are taken from Kardash (1998)).

$$R = 0.3m, \quad m = 150kg, \quad \rho = 1kg/m^3, \quad d_t = 48N\left(\frac{s}{m}\right)^2, \quad d_1 = 80Ns^2/m, \quad and \quad d_2 = 30Ns$$
$$K_{o_1} = I, \quad P_o = I, \quad K_p = 10I, \quad K_d = 0.5I$$

6 Conclusions

In this paper, we have addressed the problem of output feedback tracking control of a class of Euler-Lagrange systems subject to nonlinear dissipative loads. By imposing a monotone damping condition on the nonlinearities of the unmeasured states, the common restriction that the nonlinearities be globally Lipschitz is removed. The proposed observer-controller scheme renders the origin of the error dynamics uniformly globally asymptotically stable, in the general case. Under certain additional assumptions, the result continue to hold for a simplified control law that is less sensitive to noise and unmodeled phenomena.

7 Acknowledgements

This work was supported by the Research Council of Norway, the National Science Foundation under grant ECS-9812346, and the Air Force Office of Scientific Research under grant F49620-95-1-0409. The first and third authors would like to thank Dr. Anton Shiriaev and Dr. Antonio Loria for valuable discussions.

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