A Theorem for UGAS and ULES of (Passive) Nonautonomous Systems: Robust Control of Mechanical Systems and Ships

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Abstract

The main contribution of this paper is a theorem to guarantee *uniform* global asymptotic stability (UGAS) and uniform local exponential stability (ULES) for a class of nonlinear nonautonomous systems which includes *passive* systems. These properties (and a uniform local Lipschitz condition) guarantee robustness of stability while weaker properties, like uniform global stability plus global convergence, do not. Our main result is then used in the tracking control problem of mechanical systems and ships. We use an adaptive backstepping design and prove UGAS of the closed-loop tracking error system, in particular we obtain that both the tracking and parameter estimation errors converge uniformly globally to zero.

Notation: $\|\cdot\|$ stands for the Euclidean norm of vectors and induced norm of matrices. $\|\cdot\|_\infty$ denotes the $\mathcal{L}_\infty$ norm. We denote by B_r the set $B_r := \{x \in \mathbb{R}^{n_1} : \|x\| \leq r\}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$, $\alpha \in \mathcal{K}$, if $\alpha(\cdot)$ is strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if in addition $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\beta(\cdot,\cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t) \to 0$ as $t \to \infty$ for each $s \geq 0$. Unless otherwise specified we use in general the letter c to denote a positive constant. For positive definite matrices we use the bounds $p_m I \leq P \leq p_M I$.

1 Introduction

When designing industrial control systems it is important to include integral action in the control law in order to compensate for slowly-varying and constant disturbances. This is necessary to avoid steady-state errors both in regulation and tracking. For nonlinear systems often this is done in an *ad-hoc* manner with no proof of stability or convergence.

This paper discusses a method for integral action when backstepping. The integral part of the controller is provided by using adaptive backstepping ([10]) under the assumption of constant disturbances. These are estimated on-line by using parameter adaptation. The

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resulting error dynamics consists of the tracking error states and the parameter estimation error states. This leads to a new system of higher order than the original system. Moreover, the closed-loop system is typically time-varying since the nonlinearities, which originally depended on the state of the system, must be rewritten in terms of the tracking error states and a time-dependent reference signal.

There are various types of asymptotic stability that can be pursued for time-varying nonlinear systems. The most useful of these, from a robustness point of view, are *uniform* (global) asymptotic stability and *uniform* (local) exponential stability. These are defined for the ordinary differential equation

$$\dot{x} = f(x,t) \qquad x(t_\circ) = x_\circ \, , \tag{1}$$

having solutions denoted $x(\cdot, t_\circ, x_\circ)$, as follows:

Definition 1 *The origin of the system (1) is said to be uniformly globally asymptotically stable (UGAS) if*

1. *there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $(x_\circ, t_\circ) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ and all $t \geq t_\circ$, we have*

$$\|x(t, t_\circ, x_\circ)\| \leq \gamma(\|x_\circ\|) \; ; \tag{2}$$

2. *for each pair of strictly positive real numbers (r, σ) there exists a positive real number T such that*

$$\|x_\circ\| < r \implies \|x(t, t_\circ, x_\circ)\| \leq \sigma \quad \forall \, t \geq t_\circ + T \; . \tag{3}$$

□

Definition 2 *The origin of the system (1) is said to be uniformly locally exponentially stable (ULES) if there exist strictly positive real numbers r, k, λ such that*

$$\|x_\circ\| < r \implies \|x(t, t_\circ, x_\circ)\| \leq k \|x_\circ\| e^{-\lambda(t - t_\circ)} \; . \tag{4}$$

□

The reason these types of stability are most useful is that (at least when $f(\cdot, t)$ is locally Lipschitz uniformly in t) they guarantee total or robust stability. This is not necessarily the case for weaker forms of asymptotic stability for time-varying systems. To see this, consider the following.

Example 1 (Nonrobust GAS system) Consider a system of the form

$$\dot{x} = -a(t) x^3 \; . \tag{5}$$

When $a(t)$ is such that the origin of this system is uniformly locally asymptotically stable and $a(t)$ is bounded, it is well-known (see [7, Lemma 5.4]) that the system

$$\dot{x} = -a(t) x^3 + d(t) \tag{6}$$

is locally input-to-state stable; in particular, small bounded signals $d(t)$ and small initial conditions $x_\circ$ yield small state trajectories for all $t \geq 0$, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\max\{\|x_\circ\|, \|d\|_\infty\} \leq \delta \qquad \Longrightarrow \qquad \|x(t, t_\circ, x_\circ, d)\| \leq \epsilon \qquad \forall t \geq t_\circ \geq 0 \; . \tag{7}$$

On the other hand, consider the case where $a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is continuous, strictly decreasing with $\lim_{t\to\infty} a(t) = 0$ but with $a(t)$ nonintegrable (for example $a(t) = 1/1 + t$) i.e., $\lim_{t\to\infty} \int_0^t a(\tau)d\tau = \infty$. In this case it is easy to see that the first part of the (UGAS) definition is satisfied for the system (5). Moreover, it can be shown that for all $(x_\circ, t_\circ) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ the trajectories of (5) satisfy $\lim_{t\to\infty} \|x(t, t_\circ, x_\circ)\| = 0$. Yet, for this $a(t)$, the system (6) is not locally input-to-state stable. Indeed, let $\epsilon > 0$ be given and suppose there exists a $\delta > 0$ such that (7) holds. Let $T > 0$ be such that $\|a(t)x^3\| \leq 0.5\delta$ for all $\|x\| \leq \epsilon$ and all $t \geq T$. Let $x_\circ = \delta$ and pick $d(t) = 0$ for all $t \in [0, T]$ and $d(t) = \min\{\delta, \|a(t)x^3(t) + 0.5\delta\|\}$ for all $t > T$. By assumption $\|x(t)\| \leq \epsilon$ for all $t \geq 0$. Also $x(t) \geq 0$ for all $t \geq 0$. Hence, by construction we have $\dot{x} = 0.5\delta$ for all $t > T$. But this contradicts $\|x(t)\| \leq \epsilon$ for all $t \geq 0$. □

In this paper we contribute a theorem that states sufficient conditions for UGAS and ULES for time-varying nonlinear systems in a form that is common in nonlinear adaptive control. Our sufficient conditions are stronger than typical persistency of excitation conditions in adaptive control (see, for example, [6] and [9] where a new notion of PE for nonlinear systems was introduced) but cover the problems that we are interested in for this paper. Our conditions admit a direct Lyapunov proof for UGAS and ULES instead of having to appeal to linear systems theory and the notion of *uniform complete observability.* Besides, we have not been able to locate any results in the literature that convincingly prove *uniform* convergence, and hence UGAS, under persistency of excitation. Instead, the usual proofs simply establish trajectory by trajectory convergence (see[1] [13, Proposition 1]) or exponential convergence (for example, see[2] [19, Lemma A.5] or [7, Theorem 13.3]).

Recall that our stated reason for pursuing UGAS and ULES is robustness and/or input-to-state stability with respect to unmodeled disturbances when using certain controllers with integral action. It is worth noting that in [5] the authors solved a similar (global) input-to-*output* stability problem without appealing to UGAS by explicitly modeling the location of these *unmodeled disturbances* and further modifying the control law. In this case it was also assumed that the constant disturbances belonged to a known compact set.

The rest of our paper is organized as follows: in the next section we present our contribution on UGAS and ULES for a class of time-varying nonlinear systems. In Section 3, a mass-damper-spring system is used to demonstrate the adaptive backstepping algorithm and the utility of our stability theorem for the cases of *matching* and *extended matching* of the disturbance and the control input. Finally, a nonlinear ship tracking control problem is discussed where the tracking error dynamics is non-autonomous due to forward speed variations of the ship.

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[1] The cited proposition claims UGAS but the proof relies on a lemma that pays no attention to uniformity of convergence in the initial time.

[2] Both of the cited results prove exponential convergence by associating to each trajectory a time-varying linear system that is shown to be ULES. No attention is paid to whether the convergence is uniform over the family of linear systems generated by all trajectories starting in a compact set. Only the result in [7] claims ULES.

2 Main Result

The analysis problem we will run into in this paper is to establish uniform global asymptotic stability (UGAS) and uniform local exponential stability (ULES) for a system in the form:

$$\dot{x}_1 = h(x_1,t) + G(x,t)x_2 \tag{8a}$$

$$\dot{x}_2 = -PG(x,t)^\top \left(\frac{\partial W(x_1,t)}{\partial x_1}\right)^\top, \qquad P = P^\top > 0 \tag{8b}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ and $W : \mathbb{R}^{n_1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{C}^1$ function satisfying certain properties (see *A2* below). We assume further that all the functions are such that the solutions exist and are unique (for instance, if they are locally Lipschitz in x, uniformly in t, continuous in both arguments). The assumptions in the following theorem will be used throughout the paper to establish UGAS and ULES when designing backstepping controllers with integral action, for mechanical systems.

Theorem 1 *If Assumptions A1 and A2 below hold, then the origin of the system (8) is UGAS.*

A1 Define $G_\circ(x_2,t) := G(x,t)\,|_{x_1\equiv 0}$. Assume that there exist continuous nondecreasing functions $\rho_j : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $(j = 1,2,3)$ such that, for all $t \geq 0$, $x \in \mathbb{R}^{n_1+n_2}$

$$\max\left\{\|h(x_1,t)\|, \left\|\frac{\partial W(x_1,t)}{\partial x_1}\right\|\right\} \leq \rho_1(\|x_1\|)\,\|x_1\| \tag{9}$$

$$\max\{\|G(x,t)\|\;, \|G_\circ(x_2,t)\|\} \leq \rho_2(\|x\|) \tag{10}$$

$$\max\left\{\left\|\frac{\partial G_\circ(x_2,t)}{\partial((x_2)_i)}\right\|, \left\|\frac{\partial G_\circ(x_2,t)}{\partial t}\right\|\right\} \leq \rho_3(\|x_2\|)\;,\; i \in \{1,\ldots,n_2\}\,. \tag{11}$$

Furthermore, for each compact set $K \subset \mathbb{R}^{n_2}$ there exists $b_m > 0$ such that

$$G_\circ(x_2,t)^\top G_\circ(x_2,t) \geq b_m I \tag{12}$$

for all $(x_2,t) \in K \times \mathbb{R}_{\geq 0}$.

A2 There exist class-$\mathcal{K}_\infty$ functions α_1 and α_2 and a strictly positive real number $c > 0$ such that

$$\alpha_1(\|x_1\|) \leq W(x_1,t) \leq \alpha_2(\|x_1\|) \tag{13}$$

$$\frac{\partial W(x_1,t)}{\partial t} + \frac{\partial W(x_1,t)}{\partial x_1}h(x_1,t) \leq -c\,\|x_1\|^2\;. \tag{14}$$

Moreover, if $\alpha_2(s) \propto s^2$ for sufficiently small s then the origin is ULES. $\square$

Remark 1

- Systems of the form (8) include applications in adaptive control of linear time varying (see e.g. [6, 12, 16]) and nonlinear systems (see e.g. [7, 11]). In particular, even though the class of systems considered here includes that considered in [11, Theorem B.2.1], it is worth noticing that the condition (B.10) of the latter reference is slightly weaker than (12). As mentioned before, a further relaxation of these conditions in terms of persistency of excitation, has been reported in [15], however, for the purpose of this paper, condition (12) suffices.

- Notice also that if we define

$$F(t,x) := \begin{bmatrix} h(x_1,t) + G(x,t)x_2 \\ -PG(x,t)^\top \left(\frac{\partial W(x_1,t)}{\partial x_1}\right)^\top \end{bmatrix}$$

the system $\Sigma : v \mapsto y$, with internal dynamics $\dot{x} = F(t,x) + v(t)$, external input $v(t) := \mathrm{col}[v_1(t); 0]$ with $v_1(t) \in \mathbb{R}^{n_1}$ and output $y(t) = \frac{\partial V}{\partial x_1}(x(t))$ is output strictly passive with storage $W(x_1)$. Systems belonging to this class often include Euler-Lagrange systems (see e.g. [14].) $\square$

Proof . For uniform global stability we consider the Lyapunov function candidate $V_1 : \mathbb{R}^{n_1+n_2} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$V_1(x,t) = W(x_1,t) + \frac{1}{2}{x_2}^\top P^{-1} x_2 \tag{15}$$

which, from (13) is positive definite, radially unbounded and decrescent. The time derivative of $V_1(x,t)$ along the trajectories of (8), yields

$$\dot{V}_1(x,t) \leq -c\,\|x_1\|^2 \leq 0 \tag{16}$$

and therefore, the origin is uniformly globally stable (UGS). That is, there exists a function γ of class $\mathcal{K}_\infty$ such that $\|x(t)\| \leq \gamma(\|x_\circ\|)$, $x_\circ = x(t_\circ)$.

Now we prove global uniform attractivity, that is, point 2 in Definition 1. Notice that (8) can be rewritten as

$$\dot{x} = A(x,t)x + f(x,t) \tag{17}$$

where

$$A(x,t) = \begin{bmatrix} -I & G_\circ(x_2,t) \\ -G_\circ(x_2,t)^\top & 0 \end{bmatrix} \tag{18}$$

and

$$f(x,t) = \begin{bmatrix} h(x_1,t) + x_1 - G_\circ(x_2,t)x_2 + G(x,t)x_2 \\ -PG(x,t)^\top \left(\frac{\partial W(x_1,t)}{\partial x_1}\right)^\top + G_\circ(x_2,t)^\top x_1 \end{bmatrix}. \tag{19}$$

We will regard $f(x,t)$ as a perturbation to $\dot{x} = A(x,t)x$ and first prove that the system $\dot{x} = A(x,t)x$ is uniformly exponentially stable on each compact set of initial states. More precisely, we will show that for each $r > 0$ we can find a $\mathcal{C}^1$ function V_2 and strictly positive real numbers c_i $(i = 1,2,3)$ such that for all $(x,t) \in B_{\gamma(r)} \times \mathbb{R}_{\geq 0}$ we have $c_1\|x\|^2 \leq V_2(x,t) \leq c_2\|x\|^2$ and $\dot{V}_2 \leq -c_3\|x\|^2$. To that end we define $\omega := \gamma(r)$ and consider the Lyapunov function candidate $V_2 : B_\omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$V_2(x,t) = \frac{1}{2}\left(\|x_1\|^2 + \|x_2\|^2\right) - \varepsilon {x_1}^\top G_\circ(x_2,t)x_2 \tag{20}$$

where we impose $\varepsilon \le 1/(2\bar\rho_2)$ with $\bar\rho_2 = \rho_2(\omega)$, so that $0.25\left\|x\right\|^2 \le V_2(t,x) \le 0.75\left\|x\right\|^2$ for all $(x,t) \in B_\omega \times \mathbb{R}_{\ge 0}$. The time derivative of $V_2(x,t)$ along the trajectories of $\dot x = A(x,t)x$ is

$$\begin{aligned}\dot V_2(x,t) \;&=\; -\left\|x_1\right\|^2 - \varepsilon {x_1}^\top \overset{\cdot}{\overline{G_\circ(x_2,t)}} x_2 + \varepsilon {x_1}^\top G_\circ(x_2,t) G_\circ(x_2,t)^\top x_1 \\ &\quad + \varepsilon {x_1}^\top G_\circ(x_2,t) x_2 - \varepsilon {x_2}^\top G_\circ(x_2,t)^\top G_\circ(x_2,t) x_2 \; .\end{aligned}$$

From (10) and (11) there exists $b'_M > 0$ such that[3] $\|\overset{\cdot}{\overline{G_\circ(x_2,t)}}\| \le b'_M$ for all $(x,t) \in B_\omega \times \mathbb{R}_{\ge 0}$. Hence after some straightforward boundings we obtain that $\dot V_2(x,t)$ satisfies

$$\begin{aligned}\dot V_2(x,t) \;&\le\; -\left(\frac{1}{2} - \varepsilon\bar\rho_2^2\right)\left\|x_1\right\|^2 - \frac{\varepsilon b_m^2}{2}\left\|x_2\right\|^2 - \frac{1}{4}\left\|x_1\right\|^2 \\ &\quad -\frac{1}{2}\begin{bmatrix}\|x_1\| \\ \|x_2\|\end{bmatrix}^\top \begin{bmatrix} 1/2 & -\varepsilon(b'_M + \bar\rho_2) \\ -\varepsilon(b'_M + \bar\rho_2) & \varepsilon b_m\end{bmatrix}\begin{bmatrix}\|x_1\| \\ \|x_2\|\end{bmatrix}\end{aligned} \tag{21}$$

for all $(x,t) \in B_\omega \times \mathbb{R}_{\ge 0}$. Clearly, if

$$\varepsilon \le \frac{1}{2}\min\left\{\frac{1}{\bar\rho_2^2},\, \frac{b_m}{(b'_M + \bar\rho_2)^2}\right\} \tag{22}$$

then $\dot V_2(x,t) \le -c_3\left\|x\right\|^2$ with $c_3 = 0.25\min\{\varepsilon b_m, 1\}$.

Using the calculations above we proceed to prove global uniform attractivity for the perturbed system (17). Consider the Lyapunov function candidate $\mathcal{V} : B_\omega \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ defined by $\mathcal{V}(x,t) := \mu V_1(x,t) + V_2(x,t)$ where $\mu > 0$ is to be specified later. From (15), (20) and (13) it follows that

$$\frac{1}{4}\left\|x\right\|^2 \le \mathcal{V}(x,t) \le \alpha_3(\|x\|) \tag{23}$$

where we defined $\alpha_3(s) := \mu\alpha_2(s) + \frac{\mu}{p_m}s^2 + 0.75s^2$ and we remind the reader that $P \ge p_m I > 0$. From Assumption *A2*, (16) and (22) we have

$$\dot{\mathcal{V}}(x,t) \;\le\; -c\mu\left\|x_1\right\|^2 - c_3\left\|x\right\|^2 + \left\|\frac{\partial V_2}{\partial x}(x,t)\right\| \left\|f(x,t)\right\| \tag{24}$$

while from Assumption *A1*, there exists a continuous and nondecreasing function $\rho_4(\|x\|)$ such that

$$\max\left\{\left\|\frac{\partial V_2}{\partial x}(x,t)\right\|, \left\|f(x,t)\right\|\right\} \;\le\; \rho_4(\|x\|)\left\|x\right\| \tag{25}$$

using this bound in (24) we obtain

$$\dot{\mathcal{V}}(x,t) \;\le\; -\begin{bmatrix}\|x_1\| \\ \|x\|\end{bmatrix}\begin{bmatrix} c\mu & -0.5\bar\rho_4 \\ -0.5\bar\rho_4 & -0.5c_3\end{bmatrix}\begin{bmatrix}\|x_1\| \\ \|x\|\end{bmatrix} \;-\; 0.5c_3\left\|x\right\|^2 \tag{26}$$

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[3] With an abuse of notation let us write $\dot x_2^\top \frac{\partial G_\circ(x_2,t)^\top}{\partial x_2} := \sum_{i=1}^{n_2} (\dot x_2)_i \frac{\partial G_\circ(x_2,t)^\top}{\partial\left((x_2)_i\right)}$. Then it follows from *A1* and direct calculations that there exists a continuous nondecreasing function $\rho(s)$ such that $\left\|\overset{\cdot}{\overline{G_\circ(x_2,t)}}\right\| \le \rho(\|x\|)$.

where $\bar{\rho}_4 = \rho_4(\omega)$ that is, $\dot{\mathcal{V}}(x,t)$ is negative definite on $B_\omega \times \mathbb{R}_{\geq 0}$ if μ is sufficiently large so that $\mu \geq \bar{\rho}_4^2/(2cc_3)$. From the last term on the right hand side of (26) we obtain that $\dot{\mathcal{V}}(t) \leq -\alpha_4(\mathcal{V}(t))$ for all t such that $x(t) \in B_\omega$ and where we have defined $\alpha_4(s) := -0.5c_3[\alpha_3^{-1}(s)]^2$ which is clearly of class $\mathcal{K}_\infty$. Moreover, since $x(t) \in B_\omega$ for all $(x_\circ, t_\circ) \in B_r \times \mathbb{R}_{\geq 0}$, it follows that $\dot{\mathcal{V}}(t) \leq -\alpha_4(\mathcal{V}(t))$ for all $t \geq 0$ therefore, from standard comparison theorems [7, Lemma 2.5] we obtain the existence of a function $\beta \in \mathcal{KL}$ such that $\mathcal{V}(t) \leq \beta(\mathcal{V}_\circ, t - t_\circ)$ – where $\mathcal{V}_\circ := \mathcal{V}(x_\circ, t_\circ)$ – which implies, from (23), that $\|x(t)\| \leq 2\sqrt{\beta(\alpha_3(\|x_\circ\|), t - t_\circ)}$ for all $(x_\circ, t_\circ) \in B_r \times \mathbb{R}_{\geq 0}$. Therefore, given $r > 0$ and $\sigma > 0$ if we pick T such that $2\sqrt{\beta(\alpha_3(r), T)} \leq \sigma$, then (3) holds.

To complete the proof, notice that if $\alpha_2 \propto s^2$ then so is α_3, and ULES follows from standard results (see e.g. [7, Corollary 3.4]), looking at (23) and (26). ■

Remark 2 It is clear from the method of proof that this result hinges upon the ability to show that the unperturbed system $\dot{x} = A(x,t)x$ is ULES. The decomposition into a ULES and a perturbation is inspired by the results in [19]. For ULES of $\dot{x} = A(x,t)x$ we have imposed the bound (12). □

3 Robust tracking control of mechanical systems

The concept of *constant disturbance adaptation* will now be demonstrated on a nonlinear mass-damper-spring system where $x \in \mathbb{R}$ denotes the position, $v \in \mathbb{R}$ denotes the velocity and $\tau \in \mathbb{R}$ is the control input. Assume that the mass $m > 0$, the damping function $d(v) > 0, \forall v$ and spring stiffness $k(x) > 0, \forall x$. We consider two cases, depending on the state equation in which the constant disturbance $\theta \in \mathbb{R}$ appears. In order to compensate for this perturbation we add an integral action to the control law.

1) Integral action based on matching between the disturbance and the control input

$$\begin{aligned} \dot{x} &= v && \text{(27a)} \\ m\dot{v} + d(v)v + k(x)x &= \tau + \theta && \text{(27b)} \\ \dot{\theta} &= 0\,. && \text{(27c)} \end{aligned}$$

In this case the control input τ can compensate for θ directly (both terms are in the same state equation).

2) Integral action based on extended matching between the disturbance and the control input

$$\begin{aligned} \dot{x} &= v + \theta && \text{(28a)} \\ m\dot{v} + d(v)v + k(x)x &= \tau && \text{(28b)} \\ \dot{\theta} &= 0\,. && \text{(28c)} \end{aligned}$$

In the extended matching case there is a structural obstacle since the control law cannot be used to compensate for the unknown term θ directly. This is due the fact that θ and τ do not enter the same state equation. However, this problem can be solved by *adaptive backstepping*.

3.1 Direct matching of the disturbance and control input

Let $x_d(t)$ and $v_d(t)$ be the desired position and velocity references respectively. Then, the tracking control law for the matching case can be designed by considering the position and tracking errors, $z_1 = x - x_d$ and $\dot{z}_1 = v - v_d$. Next, we introduce the new state variable

$$z_2 := v - \alpha \, , \tag{29}$$

with α, the virtual control input given by

$$\alpha = v_d - c_1 z_1 \ , \qquad c_1 > 0 \ . \tag{30}$$

Using (30) and (29) the velocity tracking error can be written as

$$\dot{z}_1 = -c_1 z_1 + z_2 \tag{31}$$

while differentiating (29) and using (27b) we have

$$\begin{aligned} \dot{z}_2 &= \dot{v} - \dot{v}_d + c_1(v - v_d) && (32) \\ m\dot{z}_2 &= \tau - d(v)v - k(x)x + \theta - m\dot{v}_d + mc_1(v - v_d) \ . && (33) \end{aligned}$$

At this point, consider the control Lyapunov function candidate $V_1(z, \tilde{\theta}) = 0.5 z_1^2 + \frac{1}{2p}\tilde{\theta}^2$ where $p > 0$ and, $\tilde{\theta} = \hat{\theta} - \theta$ is the parameter estimation error. The time derivative of $V_1(z, \tilde{\theta})$ along (31), (28c) yields

$$\dot{V}_1(z, \tilde{\theta}) = z_1 z_2 - c_1 z_1^2 + \frac{1}{p}\tilde{\theta}\dot{\hat{\theta}} \ . \tag{34}$$

Next, consider the control Lyapunov function candidate $V_2(z, \tilde{\theta}) = V_1(z_1, \tilde{\theta}) + 0.5 m z_2^2$, whose time derivative along (31) and (33) yields

$$\dot{V}_2(z, \tilde{\theta}) = z_1 z_2 - c_1 z_1^2 + \frac{1}{p}\tilde{\theta}\dot{\hat{\theta}} + z_2\left(\tau - d(v)v - k(x)x + \theta - m\dot{v}_d + mc_1(v - v_d)\right) \ . \tag{35}$$

Choosing the control and adaptation laws as

$$\begin{aligned} \tau &= d(v)\alpha + k(x)x - \hat{\theta} + m\dot{v}_d - mc_1(v - v_d) - z_1 - c_2 z_2 && (36a) \\ \dot{\hat{\theta}} &= p z_2 && (36b) \end{aligned}$$

where $\alpha = v - z_2$, we obtain

$$\dot{V}_2(z, \tilde{\theta}) = -c_1 z_1^2 - [c_2 + d(v)] z_2^2 \tag{37}$$

which is clearly negative semidefinite. Therefore, $V_2(z, \tilde{\theta})$ qualifies as a Lyapunov function for the error dynamics (27), (36) which is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1/m & -(c_2 + d(v))/m \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/m \end{bmatrix} \tilde{\theta} \tag{38}$$

$$\dot{\tilde{\theta}} = -p \begin{bmatrix} 0 & -1/m \end{bmatrix} \begin{bmatrix} z_1 \\ m z_2 \end{bmatrix} \ . \tag{39}$$

Notice that the dissipative term $d(v) = d(z_2+\alpha) = d(z_2-c_1z_1+v_d(t)) > 0, \forall v \in \mathbb{R}$ has not been canceled out in order to exploit this damping in the design. The price paid however, is that the error dynamics is *non-autonomous* and therefore the stability analysis cannot rely on invariance principles like La Salle-Krasovskii's. Moreover, since we are interested in establishing UGAS, standard results like Barbalat's Lemma, cannot be used either. Instead, we invoke Theorem 1; let $x_1 = z$, $x_2 = \tilde{\theta}$, $G(x,t)^\top = [0 \ -1/m]$ and $W(x_1) = 0.5({z_1}^2 + m{z_2}^2)$. Assumption *A2* holds true since the feedback gains $c_1 > 0$, $c_2 > 0$, and $d(v) \geq 0$. Also the first assumption is clearly satisfied since $d(v)$ is uniformly bounded and G is a constant vector satisfying $G^\top G = 1/m^2 > 0$. Therefore, the non-autonomous system (38)–(39) is UGAS. Furthermore, since for this system we can take $\alpha(\|x_1\|) = 2W(x_1)$, for all $x_1 \in \mathbb{R}^2$, UGAS and ULES follows.

3.2 Extended matching of the disturbance and the control input

In the extended matching case, backstepping is applied to overcome the structural obstacle (see e.g. [10]). As in the previous section, $z_1 = x - x_d$ denotes the tracking error. However, notice from (28a) that in this case the velocity error is $\dot{z}_1 = \dot{x} - \dot{x}_d = v + \theta - v_d$. Next, for the state variable $z_2 = v - \alpha$ we redefine the virtual control law as $\alpha = v_d - c_1z_1 - \hat{\theta}$ so that the error dynamics becomes

$$\begin{aligned} \dot{z}_1 &= -c_1z_1 - \tilde{\theta} + z_2 && (40)\\ \dot{z}_2 &= \dot{v} - \dot{v}_d + c_1(v - v_d) + c_1(\hat{\theta} - \tilde{\theta}) + \dot{\hat{\theta}} \ . && (41)\end{aligned}$$

At this point, consider the control Lyapunov function candidate $V_1(z,\tilde{\theta}) = 0.5z_1^2 + \frac{1}{2p}\tilde{\theta}^2$, with $p > 0$, its time derivative along (40) yields

$$\dot{V}_1(z,\tilde{\theta}) = z_1z_2 - c_1z_1^2 + \tilde{\theta}\left(\frac{1}{p}\dot{\hat{\theta}} - z_1\right) \ . \tag{42}$$

We postpone the choice of update law for $\hat{\theta}$ to the next step. Consider now the control Lyapunov function $V_2(z,\tilde{\theta}) = V_1(z,\tilde{\theta}) + 0.5mz_2^2$, whose time derivative along (41) is

$$\begin{aligned} \dot{V}_2(z,\tilde{\theta}) \ = \ & z_1z_2 - c_1z_1^2 + \tilde{\theta}\left(\frac{1}{p}\dot{\hat{\theta}} - z_1 - mc_1z_2\right) + \\ & z_2\left(\tau - d(v)v - k(x)x - m\dot{v}_d + mc_1(v - v_d) + mc_1\hat{\theta} + m\dot{\hat{\theta}}\right) \ . \end{aligned} \tag{43}$$

Choosing the parameter update, and control laws as

$$\dot{\hat{\theta}} = p(z_1 + mc_1z_2) \tag{44}$$

$$\tau = d(v)\alpha + k(x)x + m\dot{v}_d - mc_1(v - v_d) - mc_1\hat{\theta} - (mp + 1)z_1 - (m^2pc_1 + c_2)z_2 \tag{45}$$

where we recall that $\alpha = v - z_2$, we obtain

$$\dot{V}_2(z,\tilde{\theta}) = -c_1z_1^2 - [c_2 + d(v)]z_2^2 \ . \tag{46}$$

which is clearly negative semi-definite. Therefore $V_2(z,\tilde{\theta})$ as defined below (42), qualifies as a Lyapunov function for the error system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1/m & -(c_2 + d(v))/m \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -c_1 \end{bmatrix} \tilde{\theta} \tag{47}$$

$$\dot{\tilde{\theta}} = -p \begin{bmatrix} -1 & -c_1 \end{bmatrix} \begin{bmatrix} z_1 \\ mz_2 \end{bmatrix} \ . \tag{48}$$

where the feedback gains $c_1 > 0$ and $c_2 > 0$. To conclude UGAS of the closed loop system above we invoke Theorem 1. The system (47), (48) differs from (38), (39) only in the definition of $G^\top = \text{col}[-1, -c_1]$. Hence it is clear that all assumptions of Theorem 1 hold true for this system as well and UGAS and ULES follows along the same lines as in the previous section.

4 Vectorial Backstepping of Ships in 3 DOF

Conventional ship control systems are designed under the assumption that the kinematic and dynamic equations of motion can be linearized so that gain-scheduling techniques and optimal control theory can be applied [2]. This is not a good assumption for tracking applications where the surge and sway positions (x, y) and yaw angle ψ must be controlled simultaneously, see [4, 1]. The main reason for this, is that the rotation matrix in yaw $J(\eta)$, typically must be linearized about 36 operating points (steps of 10 degrees) to cover the whole circle arc with adequate accuracy. In addition to this, assumptions like linear damping and negligible Coriolis and centripetal forces are only good for low-speed applications, that is station-keeping and dynamic positioning (DP). These limitations motivate us to seek for a nonlinear control design. In particular, backstepping can be used for this purpose by exploiting nonlinear system properties like symmetry of the inertia matrix, dissipative damping and skew-symmetry of the Coriolis and centripetal matrix, see [2, 3]. See also [8] for a separation principle, passivity-based output feedback design.

Moreover, when designing ship control systems, an integral action is needed in order to compensate for constant (or slowly-varying) environmental disturbances[4] due to slowly-varying ocean currents, 2nd-order wave-induced drift forces and slowly-varying wind forces.

4.1 The model

In this section we consider surface ships affected by constant environmental forces $e \in \mathbb{R}^3$, that is according to [2],

$$\begin{aligned} \dot{\eta} &= J(\eta)\nu && \text{(49a)} \\ M\dot{\nu} + C(\nu)\nu + D(\nu)\nu &= Bu + J(\eta)^\top e \,. && \text{(49b)} \end{aligned}$$

This model describes the motion of a surface ship in $n{=}3$ degrees of freedom (DOF) where $\nu = [u, v, r]^\top$ is the velocity vector decomposed in the body-fixed reference frame, $\eta = [x, y, \psi]^\top$ is the position/attitude vector decomposed in Earth-fixed coordinates and $u \in \mathbb{R}^r$ is a vector of control inputs (azimuth thrusters, main propellers and tunnel thrusters). It is assumed that $B \in \mathbb{R}^{3\times r}$ and $r \geq 3$ such that the ship is fully actuated or overactuated. According to [2] the model (49) possesses the following properties.

(i) $M = M^\top$ is positive definite, i.e., $x^\top M x > 0,\ \forall\, x \in \mathbb{R}^3 \backslash \{0\}$

(ii) $C(\nu) = -C^\top(\nu)$ is skew symmetric, i.e., $x^\top C(\nu) x = 0, \forall x \in \mathbb{R}^3$

(iii) $D(\nu)$ is strictly positive, i.e., $x^\top D(\nu) x = \frac{1}{2} x^\top [D(\nu) + D^\top(\nu)] x > 0,\ \forall x \in \mathbb{R}^3 \backslash \{0\}$

(iv) $BB^\top$ is non-singular

(v) $J(\eta)$ is the elementary rotation matrix in yaw, hence $J^\top(\eta) J(\eta) = I$

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[4] See [2] for details.

As it can be appreciated from (49), this ship model is similar to the mass-spring-damper system of Section 3, except for the coordinates transformation given by the ship kinematics equation (49a). That is, the velocities ν of the ship do not correspond exactly to the time derivative of the positions η due to the different coordinate frames in which these quantities are expressed. However, the orthogonality property (v) of the Jacobian $J(\eta)$, allows to make the following global nonlinear change of coordinates. Differentiate (49a) once and substitute the resulting expression for $\dot{\nu}$, as well as for ν, in (49b). Next, premultiply on both sides of the resulting matrix equation, by $J(\eta)$. The marine vehicle dynamics (49) is then equivalent to

$$M_\eta(\eta)\ddot{\eta} + C_\eta(\eta,\nu)\dot{\eta} + D_\eta(\eta,\nu)\dot{\eta} = J(\eta)Bu + e \tag{50}$$

where we have defined the *Earth-fixed* model matrices:

$$\begin{aligned} M_\eta(\eta) &= J(\eta)MJ^\top(\eta) \\ C_\eta(\eta,\nu) &= J(\eta)[C(\nu) - MJ^\top(\eta)\dot{J}(\eta, J(\eta)\nu)]J^\top(\eta) \\ D_\eta(\eta,\nu) &= J(\eta)D(\nu)J^\top(\eta) \ . \end{aligned}$$

Due to the orthogonality of $J(\eta)$ the dynamic model (50) possesses similar properties to those stressed for (49): (see [2])

(vi) $M_\eta(\eta) = M_\eta(\eta)^\top$ is positive definite and moreover $\exists\, m_M,\ m_m > 0$ such that $m_m I \leq M_\eta(\eta) \leq m_M I$ for all $\eta \in \mathbb{R}^3$.
(vii) The matrix $N_\eta(\eta,\nu) := 0.5\dot{M}_\eta(\eta,\nu) - C_\eta(\eta,\nu)$ is skew symmetric.
(viii) $D_\eta(\eta,\nu)$ is strictly positive.

4.2 Control design

The *control objective* is to track a smooth bounded reference trajectory given by $\ddot{\eta}_d, \dot{\eta}_d$ and η_d while rejecting the environmental disturbances e. This problem can be solved by using nonlinear coordinate transformation defined above. Furthermore, let us define the two *virtual reference trajectories* in the Earth-fixed and body-fixed reference frames:

$$\dot{\eta}_r \triangleq \dot{\eta}_d - \Lambda\tilde{\eta} \tag{51}$$

$$\nu_r \triangleq J^\top(\eta)\dot{\eta}_r \tag{52}$$

where $\tilde{\eta} = \eta - \eta_d$ is the Earth-fixed tracking error and $\Lambda > 0$ is a diagonal design matrix. Furthermore, let $s \triangleq \dot{\eta} - \dot{\eta}_r = \dot{\tilde{\eta}} + \Lambda\tilde{\eta}$ denote a measure of tracking error. In these new coordinates, the marine vehicle Earth-fixed dynamics (50) takes the form

$$M_\eta(\eta)\dot{s} = -C_\eta(\eta,\nu)s - D_\eta(\eta,\nu)s + J(\eta)[Bu - M\dot{\nu}_r - C(\nu)\nu_r - D(\nu)\nu_r + J(\eta)^\top e] \ . \tag{53}$$

At this point, let us define the *virtual control vector*

$$\dot{\eta} = J(\eta)\nu \triangleq s + \alpha_1 \tag{54}$$

where α_1 is a *stabilizing vector field* which can be chosen as $\alpha_1 = \dot{\eta}_r = \dot{\eta}_d - \Lambda\tilde{\eta}$, hence (54) becomes

$$\dot{\tilde{\eta}} = -\Lambda\tilde{\eta} + s \ . \tag{55}$$

Consider now the control Lyapunov function candidate $V_1(\tilde{\eta}) = 0.5\tilde{\eta}^\top K_p \tilde{\eta}$ where $K_p = K_p^\top > 0$ is a design matrix. The time derivative of $V_1(\tilde{\eta}) = 0.5\tilde{\eta}^\top K_p \tilde{\eta}$ along (55) yields $\dot{V}_1(\tilde{\eta}) = -\tilde{\eta}^\top K_p \Lambda \tilde{\eta} + \tilde{\eta}^\top K_p s$. Next, motivated by [17, 18] let $V_2(\tilde{\eta}, s, \eta_d(t)) = V_1(\tilde{\eta}) + 0.5 s^\top M_\eta(\eta) s$, its time derivative along (55) yields

$$\dot{V}_2(\tilde{\eta}, s, \eta_d(t)) = -\tilde{\eta}^\top K_p \Lambda \tilde{\eta} + s^\top [K_p \tilde{\eta} + M_\eta(\eta)\dot{s} + \frac{1}{2}\dot{M}_\eta(\eta, \nu) s] \ . \tag{56}$$

Substituting (53) in (56) we obtain

$$\dot{V}_2(\tilde{\eta}, s, \eta_d(t)) = -\tilde{\eta}^\top K_p \Lambda \tilde{\eta} - s^\top D_\eta(\eta, \nu) s + s^\top e \tag{57}$$

$$+ s^\top J(\eta)[J^\top(\eta) K_p \tilde{\eta} + Bu - M\dot{\nu}_r - C(\nu)\nu_r - D(\nu)\nu_r] \tag{58}$$

where we have used the properties (*vi*)-(*viii*). Hence by defining the control law

$$u = B^\dagger [M\dot{\nu}_r + C(\nu)\nu_r + D(\nu)\nu_r - J^\top(\eta) K_d s - J^\top(\eta) K_p \tilde{\eta}] \tag{59}$$

where $B^\dagger = B^\top (BB^\top)^{-1}$ we obtain

$$\dot{V}_2 = -\tilde{\eta}^\top K_p \Lambda \tilde{\eta} - s^\top (D_\eta(\eta, \nu) + K_d) s + s^\top e \ . \tag{60}$$

Clearly, in the absence of disturbances (i.e., if $e \equiv 0$) or in the case when these can be exactly canceled in the control law, $\dot{V}_2(\tilde{\eta}, s, \eta_d(t))$ is negative definite. From this, UGES can be concluded. Since in common practice this assumption is unrealistic, consider instead the augmented controller

$$u = B^\dagger [M\dot{\nu}_r + C(\nu)\nu_r + D(\nu)\nu_r - J^\top(\eta) K_d s - J^\top(\eta) K_p \tilde{\eta} - J(\eta)^\top \hat{e}] \tag{61a}$$

$$\dot{\hat{e}} = \gamma s \tag{61b}$$

where $\hat{e}$ is the estimate of e. Defining $\tilde{e} := \hat{e} - e$, the error dynamics (49), (61) in the Earth-fixed coordinates yields

$$\begin{bmatrix} \dot{\tilde{\eta}} \\ \dot{s} \end{bmatrix} = -\begin{bmatrix} \Lambda & I \\ -M_\eta^{-1}(\eta) K_p & M_\eta^{-1}(\eta)(C_\eta(\nu, \eta) + D_\eta(\eta, \nu) + K_d) \end{bmatrix} \begin{bmatrix} \tilde{\eta} \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ -M_\eta^{-1}(\eta) \end{bmatrix} \tilde{e} \tag{62a}$$

$$\dot{\tilde{e}} = \gamma s \, . \tag{62b}$$

At this point, we can invoke Theorem 1 to ensure UGAS and ULES for the closed loop system (62). For this, let $x_1 = \mathrm{col}[\tilde{\eta}, s]$, $x_2 = \tilde{e}$, $W(x_1, t) = 0.5 s^\top M_\eta(\tilde{\eta} + \eta_d(t)) s + V_1(\tilde{\eta})$, $P = \gamma I$ and $G(x_1, t) = \mathrm{col}[0; -M^{-1}(\tilde{\eta} + \eta_d(t))]$. Hence, due to the properties $(vi) - (viii)$, all conditions in the Theorem are satisfied. Indeed, notice that we can take $\alpha_2(\|x_1\|) = m_M s^2 + k_{p_M} \tilde{\eta}^2$. It is also clear that a necessary condition for this result to hold, is the orthogonality property of the Jacobian $J(\eta)$.

5 Conclusions

In this paper UGAS and ULES of nonlinear non-autonomous systems where constant disturbances are compensated for by using an integral controller have been discussed. Integral action is provided by using adaptive backstepping. The main result of the paper is a theorem for UGAS/ULES which is intended as a design tool when designing industrial controllers with integral action. Emphasis is placed on non-autonomous systems since these cannot be analyzed by using LaSalle's theorem for invariant manifolds. Moreover, for time-varying systems, arguments based on Barbalat's Lemma as often used in the Robotics literature, do not lead to uniform asymptotic stability results. We presented some applications to backstepping tracking control of ship and mechanical systems, thereby highlight the utility of our main theorem in establishing UGAS and ULES of control systems with integral action.

References

[1] M. F. Aarset, J. P. Strand, and T. I. Fossen. Nonlinear vectorial observer backstepping with integral action and wave filtering for ships. In Proc. of the IFAC Conference on Control Applications in Marine Systems (CAMS'98), pages 83–89, Fukoka, Japan, 1998.

[2] T. I. Fossen. *Guidance and control of ocean vehicles.* John Wiley & Sons Ltd., 1994.

[3] T. I. Fossen and O. E. Fjellstad. Nonlinear modelling of marine vehicles in 6 degrees of freedom. *Int. J. of Mathematical Modelling of Systems*, JMMS-1(1):17–28, 1995.

[4] T. I. Fossen and Å . Grøvlen. Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping. *IEEE Trans. Contr. Syst. Technol.*, 6(1):121–128, 1998.

[5] R. A. Freeman and P. V. Kokotović. *Robust Nonlinear control design: State-space and Lyapunov control techniques.* Birkhäuser, Boston, 1996.

[6] P. Ioannou and J. Sun. *Robust adaptive control.* Prentice Hall, New Jersey, USA, 1996.

[7] H. Khalil. *Nonlinear systems.* Macmillan Publishing Co., 2nd ed., New York, 1996.

[8] A. Loría, T. I. Fossen, and E. Panteley. Cascaded-based global output-feedback dynamic positioning of marine vessels. *IEEE Trans. Contr. Syst. Technol.*, 8(2):332–344, 2000.

[9] A. Loría, E. Panteley, and A. Teel. A new persistency-of-excitation condition for UGAS of NLTV systems: Application to stabilization of nonholonomic systems. In *Proc. 5th. European Contr. Conf.*, 1999. paper no. 500.

[10] I. Kanellakopoulos M. Krstić and P. Kokotović. *Nonlinear and Adaptive control design.* John Wiley & Sons, Inc., New York, 1995.

[11] R. Marino and P. Tomei. Global adaptive output feedback control of nonlinear systems. Part I : Linear parameterization. *IEEE Trans. on Automat. Contr.*, 38:17–32, 1993.

[12] K. S. Narendra and A. M. Anaswamy. *Stable adaptive systems.* Prentice-Hall, Inc., New Jersey, 1989.

[13] R. Ortega and A. L. Fradkov. Asymptotic stability of a class of adaptive systems. *Int. J. Adapt. Contr. Sign. Process.*, 7:255–260, 1993.

[14] R. Ortega, A. Loría, P. J. Nicklasson, and H. Sira-Ramírez. *Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications.* Comunications and Control Engineering. Springer Verlag, London, 1998. ISBN 1-85233-016-3.

[15] E. Panteley, A. Loría, and A. Teel. UGAS of NLTV systems: Applications to adaptive control. Technical Report 99-160, Lab. d'Automatique de Grenoble, UMR 5528, CNRS, France, 1999.

[16] S. Sastry and M. Bodson. *Adaptive control: Stability, convergence and robustness.* Prentice Hall Intl., 1989.

[17] J. J. Slotine and W. Li. Adaptive manipulator control: a case study. *IEEE Trans. on Automat. Contr.*, AC-33:995–1003, 1988.

[18] M. W. Spong, R. Ortega, and R. Kelly. Comments on "Adaptive Manipulator Control: A Case Study". *IEEE Trans. on Automat. Contr.*, 35(6):761, 1990.

[19] Y. Zhang, P. Ioannou, and C. Chien. Parameter convergence of a new class of adaptive controllers. *IEEE Trans. on Automat. Contr.*, 41(10):1489–1493, 1996.