On the Combination of Nonlinear Contracting Observers and UGES Controllers for Output Feedback

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***Abstract*— The paper presents a systematic method for design of observer-controllers in cascade. Uniform global exponential stability (UGES) of the resulting system is proven by assuming that the feedback control system is UGES and that the nonlinear observer can be designed using contracting analysis. The relationship between a globally contracting and UGES observer is derived using Lyapunov analysis and a line integral which follows from Taylor's theorem.**

I. Introduction

The main idea is to design a nonlinear contracting observer copying the plant and prove that the state estimate $\hat{x}$ converges to the state x and that $\tilde{x} = x - \hat{x} = 0$ is a uniformly globally exponentially stable (UGES) equilibrium point of the observer error dynamics. Contraction analysis (Lewis [11], Lohmiller [12], Kristiansen [10], Lohmiller and Slotine [14], Jouffroy [6]) is used as a design tool to derive the observer while Lyapunov analysis and Taylor's theorem (Abraham et *al*. [1]) are used to prove UGES.

The nonlinear observer is combined with a full state feedback controller where the state x is replaced by the estimate $\hat{x}$. The full state feedback control system is assumed to be UGES, and conditions for UGES of the resulting control system in cascade with the nonlinear observer is derived using Lyapunov analysis similar to Panteley and Loria [15], and Loria [13].

We believe that this result is highly useful from a practical point of view since it allows for systematic design of contracting observers and exponentially stable controllers. Examples to autonomous underwater vehicles (AUVs) and ship control are used to illustrate the design method.

II. Nonlinear Observer Design using Contraction

A. Contracting Systems

Contraction theory is a tool that can be used to study the stability of nonlinear system trajectories with respect to each other. The definition of contraction requires the *uniform negative definiteness u.n.d.* of the *Jacobian* of the system [14]:

$$\dot{x} = f(x,t) \tag{1}$$

Contracting behavior is determined upon the exact differential relation, that is:

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x,t)\delta x \tag{2}$$

where δx is a virtual displacement–*i.e.*, an infinitesimal displacement at fixed time and

$$F(x,t) = \frac{\partial f(x,t)}{\partial x} \tag{3}$$

is recognized as the *Jacobian* of the system. Define the local transformation:

$$\delta z = \Theta(x,t)\delta x \tag{4}$$

where $\Theta(x,t)$ is a coordinate transformation matrix such that:

$$\delta\dot{z} = \left[\dot{\Theta}(x,t) + \Theta(x,t)\frac{\partial f}{\partial x}(x,t)\right]\Theta^{-1}(x,t)\delta z \tag{5}$$

The *generalized Jacobian* of the δz-system is:

$$F_{\Theta}(x,t) = \left(\dot{\Theta}(x,t) + \Theta(x,t)\frac{\partial f}{\partial x}(x,t)\right)\Theta^{-1}(x,t) \tag{6}$$

The main definition and theorem of contraction are taken from Lohmiller and Slotine [14].

Definition 1 (Contraction Region): A region of the state space is called a contraction region with respect to a uniformly positive metric:

$$M_{\Theta}(x,t) = \Theta^{\top}(x,t)\Theta(x,t) \tag{7}$$

where $\Theta(x,t)$ is a differential coordinate transformation matrix, if equivalently:

$$F_{\Theta}(x,t) = \left(\dot{\Theta}(x,t) + \Theta(x,t)\frac{\partial f}{\partial x}(x,t)\right)\Theta^{-1}(x,t) \tag{8}$$

or:

$$F(x,t)^\top M_\Theta(x,t) + \dot{M}_\Theta(x,t) + M_\Theta(x,t)F(x,t) \quad (9)$$

is uniformly negative definite.

Theorem 1 (Exp. Convergence in Contracting Systems): Any trajectory, which starts in a ball of constant radius with respect to the metric $M(x,t)$, centered at a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to this trajectory.

Proof: see Lohmiller and Slotine [14]. ■

B. Nonlinear Observer Design

Consider a nonlinear *observable* and *controllable* system in the form:

$$\dot{x} = f(x,u,t) \quad (10)$$

$$z = h(x,t) \quad (11)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^p, z \in \mathbb{R}^m, t \in \mathbb{R}_+$ and where $f(x,u,t)$ and $h(x,t)$ are continuously differentiable functions.

A nonlinear state observer copying the dynamics (10)–(11) is:

$$\dot{\hat{x}} = f(\hat{x},u,t) + K(t)(z - h(\hat{x},t)) \quad (12)$$

where $K(t)(z - h(\hat{x},t))$ is the injection term.

Assume that the trajectory tracking controller can be written in the form:

$$u = \beta(\hat{x}, x_d(t)) \quad (13)$$

where x_d denotes the desired state vector. It is assumed that $x_d = x_d(t)$ is a smooth continuously differentiable function. The closed loop then becomes:

$$\dot{x} = f(x, \beta(\hat{x}, x_d(t)), t) \quad (14)$$

$$\dot{\hat{x}} = f(\hat{x}, \beta(\hat{x}, x_d(t)), t) + K(t)\,(z - h(\hat{x},t)) \quad (15)$$

If $\hat{x}$ is replaced by x, the plant (10) is obtained and it follows that x is a particular solution of (15) and $\hat{x}$ will converge exponentially to x if $\dot{\hat{x}}$ is contracting, that is:

$$\delta\dot{\hat{x}} = F(\hat{x},t)\delta\hat{x} \quad (16)$$

where

$$F(\hat{x},t) = \frac{\partial f(\hat{x}, \beta(\hat{x}, x_d(t)), t)}{\partial \hat{x}} - K(t)\frac{\partial h(\hat{x},t)}{\partial \hat{x}} \text{ is u.n.d.}$$

This is satisfied if $F + F^\top \le -\sigma I < 0$ since:

$$\delta V = \frac{1}{2}\delta\hat{x}^\top\delta\hat{x} \quad (17)$$

$$\delta\dot{V} = \delta\hat{x}^\top(F(\hat{x},t) + F^\top(\hat{x},t))\delta\hat{x} \le -\sigma\delta\hat{x}^\top\delta\hat{x}, \quad \forall\delta\hat{x} \neq 0, \sigma > 0 \quad (18)$$

which proves exponential convergence of δx to zero. Notice that the requirement that the nonlinear observer should be contracting for all $u = \beta(\hat{x}, x_d(t))$ means that the observer is *universal contracting* in u (Jouffroy [7]).

The same result is also obtained for the generalized Jacobian, that is $F_\Theta(x,t)$ is u.n.d. implies that the virtual displacement $\delta z = \Theta(x,t)\delta x$ converges to zero where $\Theta(x,t)$ is square nonsingular matrix and $M_\Theta(x,t) = \Theta^\top(x,t)\Theta(x,t)$ represents a symmetric and continuously differentiable metric.

C. UGES Observer Analysis

The following theorem shows that if the nonlinear observer (12) is contracting (globally convergent) w.r.t. the metric $M_\Theta(x,t) = I$, the equilibrium point $\tilde{x} = x - \hat{x} = 0$ will be uniform global exponential stable (UGES).

Theorem 2 (UGES Contracting Nonlinear Observer): The equilibrium point $\tilde{x} = x - \hat{x} = 0$ of the observer error dynamics:

$$\dot{\tilde{x}} = f(x, \beta(\hat{x}, x_d(t)), t) - f(\hat{x}, \beta(\hat{x}, x_d(t)), t) - K(t)(h(x,t) - h(\hat{x},t))$$

corresponding to (10)–(11) and (12) is UGES if:

$$\bar{F}(t) - K(t)\bar{H}(t) \text{ is u.n.d.} \quad (19)$$

where $K(t)$ is a bounded gain matrix and:

$$\bar{F}(t) = \int_0^1 \frac{\partial f(y(\alpha), \beta(y(\alpha), x_d(t)), t)}{\partial y} d\alpha \quad (20)$$

$$\bar{H}(t) = \int_0^1 \frac{\partial h(y(\alpha), t)}{\partial y} d\alpha \quad (21)$$

where $y(\alpha)$ is a straight line between x and $\hat{x}$ such that:

$$y(\alpha) = \alpha x + (1-\alpha)\hat{x}, \quad \alpha \in [0,1] \quad (22)$$

$$y(0) = \hat{x}, \quad y(1) = x \quad (23)$$

$$\frac{\partial y(\alpha)}{\partial \alpha} = x - \hat{x} = \tilde{x} \quad (24)$$

Proof: Taylor's theorem gives; see Abraham et *al.* [1], pp.87-88:

$$f(x, \beta(\hat{x}, x_d(t)), t) - f(\hat{x}, \beta(\hat{x}, x_d(t)), t) = \int_0^1 \frac{df(y(\alpha), \beta(y(\alpha), x_d(t)), t)}{d\alpha} d\alpha \quad (25)$$

Using the fact that:

$$\frac{df(\cdot)}{d\alpha} = \frac{\partial f(\cdot)}{\partial y}\frac{\partial y}{\partial \alpha} = \frac{\partial f(\cdot)}{\partial y}\tilde{x} \quad (26)$$

gives,

$$\int_0^1 \frac{df(y(\alpha), \beta(y(\alpha), x_d(t)), t)}{d\alpha} d\alpha = \underbrace{\left(\int_0^1 \frac{\partial f(y(\alpha), \beta(y(\alpha), x_d(t)), t)}{\partial y} d\alpha\right)}_{\bar{F}(t)} \tilde{x}$$

Similarly:

$$h(x,t) - h(\hat{x},t) = \underbrace{\left(\int_0^1 \frac{\partial h(y(\alpha),t)}{\partial y} d\alpha \right)}_{\bar{H}(t)} \tilde{x} \quad (27)$$

Then it follows from:

$$V_{obs} = \tilde{x}^\top \tilde{x} \quad (28)$$

$$\begin{aligned} \dot{V}_{obs} &= \tilde{x}^\top (\bar{F}(t) - K(t)\bar{H}(t))\tilde{x} \\ &\quad + \tilde{x}^\top (\bar{F}(t) - K(t)\bar{H}(t))^\top \tilde{x} \\ &\leq -\lambda_q \tilde{x}^\top \tilde{x} \\ &< 0, \quad \text{if } \tilde{x} \neq 0 \end{aligned} \quad (29)$$

for some $\lambda_q > 0$ since:

$$\left[\bar{F}(t) - K(t)\bar{H}(t)\right] + \left[\bar{F}(t) - K(t)\bar{H}(t)\right]^\top < 0, \quad \forall t \quad (30)$$

Consequently, V_{obs} converges exponentially to zero and the equilibrium point $\tilde{x} = x - \hat{x} = 0$ is UGES. ■

Remark 1: The condition

$$\bar{F}(t) - K(t)\bar{H}(t) \text{ is u.n.d.} \quad (31)$$

is automatically satisfied if the observer:

$$\dot{\hat{x}} = f(\hat{x}, u, t) + K(t)(z - h(\hat{x}, t)) \quad (32)$$

with $u = \beta(\hat{x}, x_d(t))$ is contracting w.r.t. $\Theta = I$, that is

$$\frac{\partial f(y, \beta(y, x_d(t)), t)}{\partial y} - K(t)\frac{\partial h(y,t)}{\partial y} \text{ is u.n.d.} \quad (33)$$

Example 1 (Linear Observer): Consider the linear time-varying system:

$$\dot{x} = A(t)x + B(t)u \quad (34)$$

$$z = C(t)x \quad (35)$$

with feedback controller

$$u = G(t)\hat{x} \quad (36)$$

where the state estimate is given by:

$$\dot{\hat{x}} = A(t)\hat{x} + B(t)G(t)\hat{x} + K(t)(z - C(t)\hat{x}) \quad (37)$$

Hence, a sufficient condition for $\hat{x}$ to be a UGES contracting observer is that:

$$F(t) = A(t) + B(t)G(t) - K(t)C(t)$$

is u.n.d.

Example 2 (Nonlinear Ship Observer): In this example we will consider the nonlinear passive observer of Fossen and Grøvlen [3] for dynamic positioning of marine vessels,

Fig. 1. Dynamic positioning of a floating production ship.

see Figure 1. The vessel model is (Fossen [4]):

$$\dot{\eta} = R(\psi)\nu \quad (38)$$

$$M\dot{\nu} + D\nu + K\eta = \tau \quad (39)$$

where $\nu = [u, v, r]^\top$ is a vector of velocities, $\eta = [x, y, \psi]^\top$ is a vector of positions and yaw angle, $K \in \mathbb{R}^{3\times3}, D \in \mathbb{R}^{3\times3}, M \in \mathbb{R}^{3\times3}$ and $R(\psi) \in SO(3)$. Hence, it follows that:

$$\dot{\nu} = A_1\eta + A_2\nu + B\tau \quad (40)$$

where:

$$A_1 = -M^{-1}K, \quad A_2 = -M^{-1}D, \quad B = M^{-1} \quad (41)$$

The matrix A_2 is Hurwitz for a ship (dissipative damping). It is assumed that only the North-East positions (x, y), and yaw angle ψ are measured, that is:

$$z = \eta = [x, y, \psi]^\top \quad (42)$$

The nonlinear observer is derived using Lyapunov theory:

$$\dot{\hat{\eta}} = R(\psi)\hat{\nu} + K_1(z - \hat{\eta}) \quad (43)$$

$$\dot{\hat{\nu}} = A_1\hat{\eta} + A_2\hat{\nu} + B\tau + K_2(z - \hat{\eta}) \quad (44)$$

The matrices K_1 and K_2 in (43) and (44) are chosen such that the observer is GES. Consider a Lyapunov function candidate:

$$V_{\text{obs}} = \frac{1}{2}\left(\tilde{\eta}^\top P_1 \tilde{\eta} + \tilde{\nu}^\top P_2 \tilde{\nu}\right) > 0, \quad \forall \tilde{\eta} \neq 0, \tilde{\nu} \neq 0 \quad (45)$$

where $P_1 = P_1^\top > 0$, $P_2 = P_2^\top > 0$, $\tilde{\eta} = \eta - \hat{\eta}$, and $\tilde{\nu} = \nu - \hat{\nu}$. Hence:

$$\dot{V}_{\text{obs}} = -\tilde{\eta}^\top Q_1 \tilde{\eta} - \tilde{\nu}^\top Q_2 \tilde{\nu} < 0, \quad \forall \tilde{\eta} \neq 0, \tilde{\nu} \neq 0 \quad (46)$$

where $Q_1 = Q_1^\top > 0$ and $Q_2 = Q_2^\top > 0$ and:

$$V_{\text{obs}}(t) \leq e^{-2\alpha t} V_{\text{obs}}(0), \quad \alpha = \lambda_{\min}(Q)/\lambda_{\max}(P) > 0 \quad (47)$$

if:

$$K_1 = P_1^{-1} Q_1 \quad (48)$$

$$K_2(\psi) = P_2^{-1} R^\top(\psi) P_1 + A_1 \quad (49)$$

where P_2 is given by:

$$\frac{1}{2}\left(P_2A_2 + A_2^{\top}P_2\right) = -Q_2 \quad (50)$$

We will now analyze the same observer using contraction theory (Jouffroy and Lottin [5]). Let:

$$\dot{\hat{\eta}} = R(\psi)\hat{\nu} + K_1(z-\hat{\eta}) := f_1(\hat{\nu},\hat{\eta},\eta) \quad (51)$$

$$\dot{\hat{\nu}} = A_1\hat{\eta} + A_2\hat{\nu} + B\tau + K_2(z-\hat{\eta}) := f_2(\hat{\nu},\hat{\eta},\eta,\tau) \quad (52)$$

The virtual dynamics of this system with respect to $(\hat{\eta},\hat{\nu})$ are:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -K_1 & R(\psi) \\ A_1 - K_2 & A_2 \end{bmatrix}\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \quad (53)$$

This system is contracting if $F^{\top} + F \leq -\beta I < 0$ which is satisfied for:

$$K_1 > 0, \quad K_2 = A_1 - R^{\top}(\psi) \quad (54)$$

It is seen that this design is somewhat restrictive because only $K_1 > 0$ can be tuned arbitrarily. To relax the condition on K_2, define $\delta z_i = \Theta_i \delta x_i$ $(i = 1,2)$ such that:

$$\begin{bmatrix} \delta\dot{z}_1 \\ \delta\dot{z}_2 \end{bmatrix} = \begin{bmatrix} -\Theta_1 K_1 \Theta_1^{-1} & \Theta_1 R(\psi)\Theta_2^{-1} \\ \Theta_2\left(A_1 - K_2\right)\Theta_1^{-1} & \Theta_2 A_2 \Theta_2^{-1} \end{bmatrix}\begin{bmatrix} \delta z_1 \\ \delta z_2 \end{bmatrix}$$

The conditions for this system to be contracting are ($F_{\Theta}^{\top} + F_{\Theta} \leq -\beta I < 0$):

$$-\Theta_1 K_1 \Theta_1^{-1} < 0 \quad (55)$$

$$\Theta_2 A_2 \Theta_2^{-1} < 0 \quad (56)$$

$$-\left(\Theta_2\left(A_1 - K_2\right)\Theta_1^{-1}\right)^{\top} = \Theta_1 R(\psi)\Theta_2^{-1} \quad (57)$$

Finally, choosing $P_i = \Theta_i^{\top}\Theta_i$ $(i = 1,2)$, yields:

$$K_1^{\top}P_1 + P_1K_1 > 0 \quad (58)$$

$$A_2^{\top}P_2 + P_2A_2 > 0 \quad (59)$$

$$K_2(\psi) = P_2^{-1}R^{\top}(\psi)P_1 + A_1 \quad (60)$$

where $K_1 > 0$. These conditions are recognized as the GES conditions (48)–(50) of the Lyapunov analysis in Fossen and Grøvlen [3]. This can also be proven by using a similar technique as in Theorem 2 with $\Theta \neq I$, see Jouffroy, Slotine and Fossen [8]. A simulation study of the nonlinear observer is found in [3] and [5].

III. UGES Cascaded Observer-Controller Design

This section shows how we can replace the state x with the estimated state $\hat{x}$ (output from a UGES contracting observer) in a nonlinear full state feedback controller and still guarantee that the closed-loop observer-controller system is UGES.

Theorem 3 (Nonlinear Observer-Controller Design):

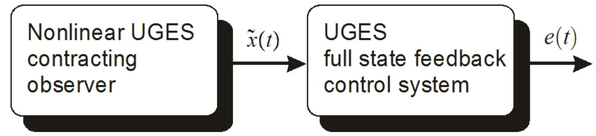

Fig. 2. Nonlinear contracting observer in cascade with a full state UGES control system.

Consider the nonlinear system

$$\dot{x} = f(x,u,t) \quad (61)$$

$$z = h(x,t) \quad (62)$$

with feedback control law

$$u = \beta(x, x_d(t)) \quad (63)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^p, z \in \mathbb{R}^m, t \in \mathbb{R}$, and where $f(x,u,t), h(x,t)$ and $\beta(x,x_d(t))$ are continuously differentiable functions. It is assumed that the desired state $x_d = x_d(t)$ is a smooth continuously differentiable function. Suppose that $e = x - x_d = 0$ is an exponential stable equilibrium point of the nominal system:

$$\dot{e} = f(x, \beta(x, x_d(t)), t) - \dot{x}_d(t) := a(e,t) \quad (64)$$

and let $V_{con}(e,t)$ be a Lyapunov function (Khalil [9]):

$$c_1\left|e\right|^2 \leq V_{con}(e,t) \leq c_2\left|e\right|^2 \quad (65)$$

$$\frac{\partial V_{con}}{\partial t} + \frac{\partial V_{con}}{\partial z}a(e,t) \leq -c_3\left|e\right|^2 \quad (66)$$

$$\left|\frac{\partial V_{con}(e,t)}{\partial e}\right| \leq c_4\left|e\right| \quad (67)$$

for some positive constants c_1, c_2, c_3 and c_4 and where $|\cdot|$ denotes the Euclidian norm. The full state feedback controller (63) can be replaced by:

$$u = \beta(\hat{x}, x_d(t)) \quad (68)$$

where $\hat{x}$ is found by integrating:

$$\dot{\hat{x}} = f(\hat{x}, \beta(\hat{x}, x_d(t)), t) + K(t)(z - h(\hat{x},t)) \quad (69)$$

Assume that $\dot{\hat{x}}$ is a UGES observer satisfying (Theorem 2):

$$\int_0^1 \frac{\partial f(y(\alpha), \beta(y(\alpha), x_d(t)), t)}{\partial y}d\alpha - K(t)\int_0^1 \frac{\partial h(y(\alpha),t)}{\partial y}d\alpha$$

is *u.n.d.* and that the perturbed system can be written:

$$\dot{e} = a(e,t) + b(e,\tilde{x},t) \quad (70)$$

where the perturbation $b(e,\tilde{x},t)$ satisfies:

$$\left|b(e,\tilde{x},t)\right| \leq \gamma_1\left|\tilde{x}\right| + \gamma_2\left|e\right|, \quad \forall e,\tilde{x},t \quad (71)$$

where γ_i $(i = 1,2)$ are two nonnegative constants satisfying:

$$\gamma_1^2 < \frac{4(c_3 - c_4\gamma_2)\lambda_q}{c_4^2}, \quad \gamma_2 < \frac{c_3}{c_4} \quad (72)$$

then the equilibrium point $(\tilde{x}, e) = (0, 0)$ of the observer-controller error dynamics is UGES.

Remark 2 (Condition $|b(e, \tilde{x}, t)| \leq \gamma_1 |\tilde{x}| + \gamma_2 |e|$): This is the critical condition from a practical point of view since it puts limitations on the growth rates of $\tilde{x}$ and e. This again put limitations on which type of physical systems that will satisfy Theorem 3.

Proof: Consider the perturbed nonlinear system:

$$\dot{e} = a(e, t) + b(e, \tilde{x}, t) \tag{73}$$

The derivative of V_{con} along the trajectories of (73) is given by:

$$\dot{V}_{con}(e, t) = \frac{\partial V_{con}}{\partial t} + \frac{\partial V_{con}}{\partial e} a(e, t) + \frac{\partial V_{con}}{\partial e} b(e, \tilde{x}, t) \tag{74}$$

The first two terms on the right-hand side of $\dot{V}_{con}$ are negative definite according to (66). The third term is the effect of the perturbation. Consequently,

$$\begin{aligned} \dot{V}_{con}(e, t) &\leq -c_3 |e|^2 + \left| \frac{\partial V_{con}}{\partial e} \right| (\gamma_1 |\tilde{x}| + \gamma_2 |e|) \\ &\leq -c_3 |e|^2 + c_4 \gamma_1 |e| |\tilde{x}| + c_4 \gamma_2 |e|^2 \end{aligned} \tag{75}$$

Hence,

$$\dot{V}_{con}(e, t) \leq -(c_3 - c_4 \gamma_2) |e|^2 + c_4 \gamma_1 |e| |\tilde{x}| \tag{76}$$

Consider the following Lyapunov function:

$$V(\tilde{x}, e, t) = V_{con}(e, t) + V_{obs}(\tilde{x}, t) \tag{77}$$

for the total system (observer and controller). Then:

$$\dot{V}(\tilde{x}, e, t) \leq -(c_3 - c_4 \gamma_2) |e|^2 + c_4 \gamma_1 |e| |\tilde{x}| - \lambda_q |\tilde{x}|^2 \tag{78}$$

where $\lambda_q > 0$ is given by (29). Hence:

$$\dot{V}(\tilde{x}, e, t) \leq -[|e|, |\tilde{x}|] \begin{bmatrix} c_3 - c_4 \gamma_2 & -\frac{1}{2} c_4 \gamma_1 \\ -\frac{1}{2} c_4 \gamma_1 & \lambda_q \end{bmatrix} \begin{bmatrix} |e| \\ |\tilde{x}| \end{bmatrix} \tag{79}$$

If γ_1 and γ_2 satisfy (72), we see that $\dot{V}(\tilde{x}, e, t) < 0, \forall \tilde{x} \neq 0, e \neq 0$ and the equilibrium point $(\tilde{x}, e) = (0, 0)$ is UGES. ■

Remark 3 (Negative semi-definite $\dot{V}(\tilde{x}, e, t)$): The case $\dot{V}(\tilde{x}, e, t) \leq 0$ (negative semi-definite) can be analyzed using a similar approach as Panteley and Loria [15], see also Loria [13]. This will give conditions for a UGAS equilibrium point $(e, \tilde{x})$ of a nonlinear UGES contracting observer in cascade with a UGAS feedback control system.

IV. Case Study

In the case study we will consider a nonlinear contracting observer in cascade with a 6 DOF autonomous underwater vehicle (AUV) controller based on backstepping. The dynamic model of ODIN (Omni-Directional Intelligent Navigator), which is the sphere-shaped, eight-thrustered AUV (Choi, Yuh and Takashige [2]), can be written in the form, see Figure 3:

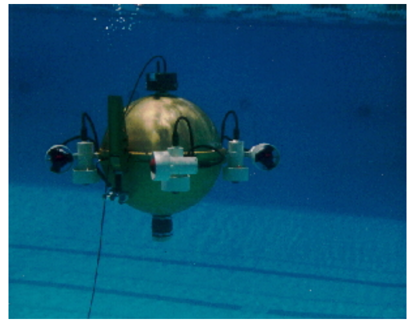

Fig. 3. The ODIN (Omni-Directional Intelligent Navigator) AUV. Courtesy to the Center for Underwater Robotic Technology, University of Hawaii.

$$\dot{\eta} = J(\eta)\nu \tag{80}$$

$$M\dot{\nu} + D(\nu)\nu + g(\eta) = \tau \tag{81}$$

where $\eta = [x, y, z, \phi, \theta, \psi]^\top$ is a vector of positions and Euler angles, $\nu = [u, v, w, p, q, r]^\top$ is a vector of body-fixed velocities and $\tau \in \mathbb{R}^6$ is the vector of generalized control forces, see Fossen [4] for details. The system inertia and damping matrices are:

$$\begin{aligned} M &= \mathrm{diag}\{m - X_{\dot{u}}, m - Y_{\dot{v}}, m - Z_{\dot{w}}, \\ &\quad I_x - K_{\dot{p}}, I_y - M_{\dot{q}}, I_z - N_{\dot{r}}\} \end{aligned} \tag{82}$$

$$\begin{aligned} D(\nu) &= \mathrm{diag}\{-X_{|u|u}|u|, -Y_{|v|v}|v|, -Z_{|w|w}|w|, \\ &\quad -K_{|p|p}|p|, -M_{|q|q}|q|, -N_{|r|r}|r|\} \end{aligned} \tag{83}$$

where the diagonal structure is due to the spherical shape of the vehicle. The transformation matrix is (see [4], pp. 23–26):

$$J(\eta) = \begin{bmatrix} R_b^n(\Theta) & 0_{3\times3} \\ 0_{3\times3} & T_\Theta(\Theta) \end{bmatrix} \tag{84}$$

where $\Theta = [\phi, \theta, \psi]^\top$ and:

$$R_b^n(\Theta) = \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & -c\psi s\phi + s\theta s\psi c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$

$$T_\Theta(\Theta) = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix}$$

For notational convenience $s\cdot = \sin(\cdot)$, $c\cdot = \cos(\cdot)$ and $t\cdot = \tan(\cdot)$. The restoring forces and moments are:

$$g(\eta) = \left[0,\ 0,\ 0,\ (z_g - z_b)Wc\theta s\phi,\ (z_g - z_b)Ws\theta,\ 0\right]^\top \tag{85}$$

where $W = mg$ is the weight of the vehicle and z_g and z_b denote the center of gravity and buoyancy, respectively (Fossen [4], pp. 75–77).

A. Nonlinear PD-Controller

1) Full state feedback: Vectorial backstepping of an AUV in 6 DOF can be done in two steps.:

Step 1: Define the *virtual control vector*:

$$\nu = s + \alpha_1 \tag{86}$$

where $s \in \mathbb{R}^6$ is a new state variable and $\alpha_1 \in \mathbb{R}^6$ is a stabilizing function chosen as:

$$\alpha_1 = \nu_r, \quad \nu_r = J^{-1}(\eta)(\dot{\eta}_d - \Lambda\tilde{\eta}) \tag{87}$$

where $\Lambda > 0$ is a diagonal design matrix and $\tilde{\eta} = \eta - \eta_d \in \mathbb{R}^6$ is the tracking error. It is assumed that $\dot{J}(\eta) = 0$ (slow motion) and $J^{-1}(\eta)$ exists for all η. This matrix is singular for $\theta = \pm 90$ deg if an *Euler angle* representation is chosen. The representation singularity can, however, be avoided by using *quaternions*. Combining (86) and (87), yields:

$$\dot{\tilde{\eta}} = -\Lambda\tilde{\eta} + J(\eta)s \tag{88}$$

Step 2: Let the controller be chosen as:

$$\tau = M\dot{\nu}_r + D(\nu)\nu_r + g(\eta) - J^\top(\eta)K_p\tilde{\eta} - K_d s \tag{89}$$

where $K_d = K_d^\top > 0$. Since $s = \nu - \nu_r$, Eqs. (81) and (89) become:

$$M\dot{s} + D(\nu)s + K_d s + J^\top(\eta)K_p\tilde{\eta} = 0 \tag{90}$$

Consider the CLF:

$$V_{con} = \frac{1}{2}s^\top M s + \frac{1}{2}\tilde{\eta}^\top K_p\tilde{\eta} > 0, \qquad \forall s \neq 0, \tilde{\eta} \neq 0 \tag{91}$$

where $M = M^\top > 0$ and $K_p = K_p^\top > 0$. Hence:

$$\begin{aligned} \dot{V}_{con} &= s^\top M\dot{s} + \tilde{\eta}^\top K_p\dot{\tilde{\eta}} \\ &= -s^\top(D(\nu) + K_d)s - \tilde{\eta}^\top K_p\Lambda\tilde{\eta} \\ &< 0, \quad \forall s \neq 0, \tilde{\eta} \neq 0 \end{aligned} \tag{92}$$

since $D(\nu) \geq 0$. Consequently, the equilibrium point $(s, \tilde{\eta}) = (0, 0)$ is UGES.

2) Output feedback: An output feedback controller using only position measurements will now be designed. Let:

$$z = \eta \tag{93}$$

be the measurement equation. The velocity vector ν is replaced by the state estimate $\hat{\nu}$ according to:

$$\tau = M\dot{\hat{\nu}}_r + D(\nu_r)\nu_r + g(z) - J^\top(z)K_p\hat{e}_1 - K_d\hat{e}_2 \tag{94}$$

where $D(\nu_r)\nu_r$ is used to compensate for $D(\nu)\nu$ and:

$$\begin{aligned} \hat{e}_1 &\triangleq \hat{\eta} - \eta_d, \quad \hat{e}_2 \triangleq \hat{s} \triangleq \hat{\nu} - \nu_r \\ \dot{\hat{\nu}}_r &\triangleq \dot{\nu}_r + J^{-1}(z)\Lambda J(z)(\nu - \hat{\nu}) \end{aligned}$$

For notational convenience, let the error terms be:

$$\tilde{x}_1 \triangleq \eta - \hat{\eta}, \quad \tilde{x}_2 \triangleq \nu - \hat{\nu}$$

such that

$$\begin{aligned} \hat{e}_1 &= \tilde{\eta} - \tilde{x}_1, \quad \hat{e}_2 = s - \tilde{x}_2 \\ \dot{\hat{\nu}}_r &= \dot{\nu}_r + J^{-1}(z)\Lambda J(z)\tilde{x}_2 \end{aligned}$$

The damping matrix satisfies the line integral, see (25):

$$D(\nu)\nu - D(\nu_r)\nu_r = \underbrace{\left(\int_0^1 \frac{\partial\left[D(y(\alpha)y(\alpha)\right]}{\partial y}d\alpha\right)}_{\bar{D}(\nu,\nu_r)}\underbrace{(\nu - \nu_r)}_{s}$$

where

$$y(\alpha) = \alpha\nu + (1-\alpha)\nu_r \in \mathbb{R}^6, \quad \alpha \in [0, 1] \tag{95}$$

Since $D(\nu)$ is a diagonal damping matrix with elements $d_i|\nu_i|\nu_i$ where $d_i > 0$ $(i = 1, ..., 6)$ it follows that:

$$\begin{aligned} \bar{D}(\nu, \nu_r) &= \int_0^1 \left(\frac{\partial}{\partial y}\left[D(y(\alpha))y(\alpha)\right]\right)d\alpha \\ &= 2\mathrm{diag}\left\{d_1\int_0^1 |y_1(\alpha)|d\alpha, ..., d_6\int_0^1 |y_6(\alpha)|d\alpha\right\} > 0 \end{aligned}$$

Hence:

$$s^\top\bar{D}(\nu, \nu_r)s > 0, \quad \forall\nu, \nu_r \tag{96}$$

Combining (80)–(81) and (94), the closed-loop system becomes:

$$\dot{\tilde{\eta}} = -\Lambda\tilde{\eta} + J(z)s \tag{97}$$

$$\begin{aligned} M\dot{s} + (\bar{D} + K_d)s + J^\top K_p\tilde{\eta} &= J^\top K_p\tilde{x}_1 \\ &+ (K_d + MJ^{-1}\Lambda J)\tilde{x}_2 \end{aligned} \tag{98}$$

or

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \dot{\tilde{\eta}} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} -\Lambda & J(z) \\ -J^\top(z)K_p & -\bar{D}(\nu,\nu_r) - K_d \end{bmatrix}\begin{bmatrix} \tilde{\eta} \\ s \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ J^\top(z)K_p & K_d + MJ^{-1}\Lambda J \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The error dynamics is in the form:

$$\dot{e} = a(e, t) + b(\tilde{x}, t) \tag{99}$$

where

$$\begin{aligned} a(e,t) &= \begin{bmatrix} -\Lambda & J(z) \\ -M^{-1}J^\top(z)K_p & -M^{-1}(\bar{D}(\nu,\nu_r) + K_d) \end{bmatrix} e \\ b(\tilde{x}, t) &= \begin{bmatrix} 0 & 0 \\ M^{-1}J^\top(z)K_p & M^{-1}K_d + J^{-1}\Lambda J \end{bmatrix}\tilde{x} \end{aligned}$$

Then it follows that:

$$\begin{aligned} |b(\tilde{x}, t)| &\leq \gamma_1|\tilde{x}|, \\ \gamma_1 &= \left\|\begin{bmatrix} 0 & 0 \\ M^{-1}K_p & M^{-1}K_d + J^{-1}\Lambda J \end{bmatrix}\right\| \end{aligned}$$

and $\gamma_2 = 0$. Then we must require that (Theorem 3):

$$\gamma_1^2 < 4c_3\lambda_q/c_4^2 \tag{100}$$

B. Nonlinear Contracting Observer

A nonlinear contracting observer copying the system dynamics is designed as:

$$\dot{\hat{\eta}} = J(z)\hat{\nu} + K_1(z - \hat{\eta}) \quad (101)$$

$$M\dot{\hat{\nu}} + D(\hat{\nu})\hat{\nu} + g(z) = \tau + J^\top(z)K_2(z - \hat{\eta}) \quad (102)$$

where $z = \eta$ and:

$$\tau = M\dot{\nu}_r + D(\nu_r)\nu_r + g(z) - J^\top(z)K_p(\hat{\eta} - \eta_d) - K_d(\hat{\nu} - \nu_r) \quad (103)$$

Let $M_o = \mathrm{diag}\{I, M\} > 0$. The Jacobian for the system (101)–(102) with respect to $(\hat{\eta}, \hat{\nu})$ under the assumption that $\tau = 0$ is:

$$F_{obs} = M_o^{-1} \begin{bmatrix} -K_1 & J(z) \\ -J^\top(z)K_2 & -\mathrm{diag}\{d_1|\hat{\nu}_1|, ..., d_6|\hat{\nu}_6|\} \end{bmatrix}$$

The output feedback controller (103) contributes with the additional terms:

$$F_{con} = M_o^{-1} \begin{bmatrix} 0 & 0 \\ -J^\top(z)K_p & -K_d - MJ^{-1}\Lambda J \end{bmatrix}$$

Consequently, if the observer and controller gain matrices are chosen as:

$$K_p > 0, \quad K_d > MJ^{-1}\Lambda J$$
$$K_1 > 0, \quad K_2 = I - K_p$$

the total Jacobian $F = F_{obs} + F_{con}$ is *u.n.d.* It then follows from Theorem 3 that the observer-controller error dynamics is UGES if K_p and K_d are chosen such that $\gamma_1^2 < 4c_3\lambda_q/c_4^2$. This guarantees that the tracking error $\tilde{\eta}$ and $\tilde{\nu}$, and observer estimation error $\tilde{x}$ converge exponentially to zero.

V. CONCLUSIONS

A nonlinear cascaded design technique for observer-controller design has been presented. The nonlinear observer was constructed using contraction analysis such that the plant dynamics can be copied directly into the observer equations. It was shown by using *Lyapunov analysis* in conjuncture with *Taylor's theorem* that the equilibrium point of the observer error dynamics was UGES. Furthermore, the nonlinear observer is combined with a UGES full state feedback controller in cascade and the resulting error dynamics was proven to be UGES under a set of conditions. The main result is quite useful from a practical point of view since it allows for a modular design of the observer and feedback control systems. Examples to AUV and ship control were used to illustrate the design method.

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