# CONTROL-ORIENTED MODELING OF A 2-BODY INTERCONNECTED MARINE STRUCTURE

**Mícheál Ó'Catháin** [*,1] **Thor I. Fossen** [**]
**Bernt J. Leira** [***]

[*] *Dept. Electronic Engineering, NUIM, IRELAND and Dept. Marine Technology, NTNU, NORWAY*
[**] *Dept. Engineering Cybernetics, NTNU, NORWAY and Centre for Ships and Ocean Structures (CeSOS)*
[***] *Dept. Marine Technology, NTNU, NORWAY*

Abstract: A model for a 2 body interconnected marine system is presented. The Newton-Euler equations with eliminated constraints (NEEC) are utilised to capture the rigid body dynamics. This approach leads to convenient integration of hydrodynamic forces acting on each body, including forces due to hydrodynamic coupling between bodies. The model is validated using data from wave-tank experiments carried out by the first author. 

Keywords: multibody dynamics, hydrodynamics, time-domain

## 1. INTRODUCTION

In the field of marine control, the formulation of the Cummins equation (Cummins, 1962) into its state-space approximation has been important. This is in order to include hydrodynamic memory effects in the equations of motion, used in control-oriented models of marine vessels and structures. This state-space approximation of the Cummins equation is the subject of a number of papers, including Yu and Falnes (1995), Kristiansen and Egeland (2003) and Jordán and Beltrán-Aguedo (2004). Application of the approach of Kristiansen and Egeland (2003) to a number of marine vessel control problems has been carried out by Fossen (2005) and Fossen and Smøgeli (2004). The Cummins equation may also be approximated by a transfer function, as is presented in McCabe *et al.* (2005). Since the Cummins equation assumes entirely linear equations of motion, both the state-space and transfer function approaches are essentially equivalent. Another approach which uses frequency-domain data to estimate the coefficients of a continuous-time transfer function from the force to the velocity of a vessel has been proposed by Perez and Lande (2006). In the current paper, the state-space approach of Kristiansen and Egeland (2003) is utilised. This gives a control-oriented representation of the hydrodynamic forces and moments acting on the two bodies, using the parameter values calculated using the multibody functionality of the 3D linear potential theory program WAMIT™.

In this paper, the rigid-body dynamics of a *two-body* wave-energy system are modeled using the Newton-Euler Equations of Motion with Eliminated Constraint Forces (NEEC), an energy based technique which builds on the method of virtual work and the D'Alembert principle (Egeland and Gravdahl, 2002). The contribution of this paper is to show how this approach leads to a convenient

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integration of all external forces, including hydrodynamic forces, acting on a multibody marine system.

The paper is organised as follows: the NEEC approach to modeling marine multibody systems is explained in Section 2. In Section 3, a specific example illustrates the application of the approach, i.e. a wave-energy system composed of 2 interconnected barges. The resulting model is validated against experimental data. Following this there is a discussion on reducing the model to a form suitable for use in controller and observer design.

## 2. THEORY

### *2.1 Constraints*

For a free floating (unconstrained) marine vessel, the number of degrees of freedom is equal to 6. The generalised Cartesian coordinates recommended by SNAME (1950) for such a free floating vessel are independent, i.e. $\boldsymbol{\eta} = [x, y, z, \phi, \theta, \psi]^\top$. If we introduce constraints between the bodies in a multibody system, the number of degrees of freedom (DOF) of the system is reduced. Some of the Cartesian (generalised) coordinates thus become redundant, so that it is possible to describe the motion of each body in the system using fewer (independent) coordinates. In other words, the number of independent coordinates is reduced due to the constraints. The *independent coordinates* of a multibody systems are also called the system *degrees of freedom* (Shabana, 2005). In this paper we will write the vector of independent variables as $\mathbf{q} \in \mathbb{R}^{n_\mathbf{q}}$, and the number of DOF as $n_\mathbf{q}$. The time derivative of the independent variables will be contained in the vector $\mathbf{s} \in \mathbb{R}^{n_\mathbf{q}}$.

### *2.2 Newton-Euler Equations of Motion with Eliminated Constraints*

Egeland and Gravdahl (2002) gives the Newton-Euler Equations of Motion with Eliminated Constraint Forces (NE-EC) as:

$$\sum_{k=1}^{N}\left[\left(\frac{\partial \mathbf{v}_{c_k}^{b_k}}{\partial \mathbf{s}}\right)^\top \left(m_k \dot{\mathbf{v}}_{c_k}^{b_k} + m_k \mathbf{S}(\boldsymbol{\omega}_{nb_k}^{b_k})\mathbf{v}_{c_k}^{b_k}\right)\right] + \sum_{k=1}^{N}\left[\left(\frac{\partial \boldsymbol{\omega}_{nb_k}^{b_k}}{\partial \mathbf{s}}\right)^\top \left(\mathbf{I}_{c_k}\dot{\boldsymbol{\omega}}_{nb_k}^{b_k} + \mathbf{S}(\boldsymbol{\omega}_{nb_k}^{b_k})\mathbf{I}_{c_k}\boldsymbol{\omega}_{nb_k}^{b_k}\right)\right] = \sum_{k=1}^{N}\left[\left(\frac{\partial \mathbf{v}_{c_k}^{b_k}}{\partial \mathbf{s}}\right)^\top \mathbf{f}_{c_k}^{b_k} + \left(\frac{\partial \boldsymbol{\omega}_{nb_k}^{b_k}}{\partial \mathbf{s}}\right)^\top \mathbf{m}_{c_k}^{b_k}\right] \tag{1}$$

where:

- $k$ is the number of the body under consideration, in a system of $N$ bodies
- $\mathbf{v}_{c_k}^{b_k}$ is the linear velocity of the center of gravity $c_k$ of body $k$, expressed in the $b_k$-frame, i.e. the body-fixed frame in body $k$.
- $\boldsymbol{\omega}_{nb_k}^{b_k}$ is the angular velocity of the body-fixed $b_k$-frame about the inertial $n$-frame, expressed in the $b_k$-frame.
- $\mathbf{f}_{c_k}^{b_k}$ and $\mathbf{m}_{c_k}^{b_k}$ are the external forces and moments acting on and about the center of gravity of each body.
- $\frac{\partial \mathbf{v}_{c_k}^{b_k}}{\partial \mathbf{s}}$ is known as the *partial linear velocity* for body $k$.
- $\frac{\partial \boldsymbol{\omega}_{nb_k}^{b_k}}{\partial \mathbf{s}}$ is known as the *partial angular velocity* for body $k$.
- $m_k$ is the mass of body $k$ and $\mathbf{I}_{c_k}$ is the inertia matrix of body $k$, about its center of gravity $c_k$.
- the skew symmetric matrix $\mathbf{S}$ is the matrix algebra equivalent to the cross product, i.e. $\mathbf{S}(\boldsymbol{\omega})\mathbf{r} = \boldsymbol{\omega} \times \boldsymbol{r}$

Our goal is to express Equation 1 in a somewhat more compact form. To this end, we begin by defining the *velocity transformation matrix* $\mathbf{P} \in \mathbb{R}^{n_\mathbf{q} \times 6N}$:

$$\mathbf{P} = \left[\left(\frac{\partial \mathbf{v}_{c_1}^{b_1}}{\partial \mathbf{s}}\right)^\top, \left(\frac{\partial \boldsymbol{\omega}_{nb_1}^{b_1}}{\partial \mathbf{s}}\right)^\top, \ldots, \ldots, \left(\frac{\partial \mathbf{v}_{c_N}^{b_N}}{\partial \mathbf{s}}\right)^\top, \left(\frac{\partial \boldsymbol{\omega}_{nb_N}^{b_N}}{\partial \mathbf{s}}\right)^\top\right] \tag{2}$$

A useful formulation results from noticing (Fossen, 2002):

$$\begin{bmatrix} m_k \dot{\mathbf{v}}_{c_k}^{b_k} + m_k \mathbf{S}(\boldsymbol{\omega}_{nb_k}^{b_k})\mathbf{v}_{c_k}^{b_k} \\ \mathbf{I}_{c_k}\dot{\boldsymbol{\omega}}_{nb_k}^{b_k} + \mathbf{S}(\boldsymbol{\omega}_{nb_k}^{b_k})\mathbf{I}_{c_k}\boldsymbol{\omega}_{nb_k}^{b_k} \end{bmatrix} = \mathbf{M}_{RB}^{b_k}\dot{\boldsymbol{\nu}}_k + \mathbf{C}_{RB}^{b_k}\boldsymbol{\nu}_k \tag{3}$$

where $\boldsymbol{\nu}_k = [\mathbf{v}_{c_k}^{b_k\,\top}, \boldsymbol{\omega}_{nb_k}^{b_k\,\top}]^\top$. In addition, $\mathbf{M}_{RB}^{b_k} \in \mathbb{R}^{6\times 6}$, the rigid-body inertia matrix of body $k$, is unique and satisfies:

$$\mathbf{M}_{RB}^{b_k} = \mathbf{M}_{RB}^{b_k\,\top} > 0, \qquad \dot{\mathbf{M}}_{RB}^{b_k} = \mathbf{0}_{6\times 6}$$

$$\mathbf{M}_{RB}^{b_k} = \begin{bmatrix} m_k \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{I}_{c_k} \end{bmatrix} \tag{4}$$

and $\mathbf{C}_{RB}^{b_k}(\boldsymbol{\nu}_k) \in \mathbb{R}^{6\times 6}$ is the non-unique Coriolis-Centripetal matrix. One instance where $\mathbf{C}_{RB}^{b_k}(\boldsymbol{\nu}_k)$ is *skew-symmetric* is as follows (Fossen, 2002):

$$\mathbf{C}_{RB}^{b_k}(\boldsymbol{\nu}_k) = \begin{bmatrix} \mathbf{0}_{3\times 3} & -m_k \mathbf{S}(\mathbf{v}_{c_k}^{b_k}) \\ -m_k \mathbf{S}(\mathbf{v}_{c_k}^{b_k}) & -\mathbf{S}(\mathbf{I}_{c_k}\boldsymbol{\omega}_{nb_k}^{b_k}) \end{bmatrix} \tag{5}$$

Further, we write the generalised velocities $\boldsymbol{\nu} = [\boldsymbol{\nu}_1^\top, ..., \boldsymbol{\nu}_N^\top]^\top$ in terms of $\mathbf{s}$, the independent velocities:

$$\boldsymbol{\nu} = \mathbf{P}^\top \mathbf{s} \tag{6}$$

and hence:

$$\dot{\boldsymbol{\nu}} = \mathbf{P}^\top \dot{\mathbf{s}} + \dot{\mathbf{P}}^\top \mathbf{s} \tag{7}$$

Drawing on Equations 1, 2, 3, 6 and 7, we arrive at the following result:

$$\mathbf{M}^g_{RB}\dot{\mathbf{s}} + \mathbf{C}^g_{RB}(\mathbf{q},\mathbf{s})\mathbf{s} = \boldsymbol{\tau}^g_{RB} \tag{8}$$

where $\mathbf{M}^g_{RB} \in \mathbb{R}^{n_\mathbf{q}\times n_\mathbf{q}}$ is the generalised mass matrix for the $n_\mathbf{q}$ DOF multibody system, given by:

$$\mathbf{M}^g_{RB} = \mathbf{P}\mathbf{M}^b_{RB}\mathbf{P}^\top \tag{9}$$

where $\mathbf{M}^b_{RB} \in \mathbb{R}^{6N\times 6N} = diag(\mathbf{M}^{b_1}_{RB}...\mathbf{M}^{b_N}_{RB})$.
Also we have defined $\mathbf{C}^g_{RB}(\mathbf{q},\mathbf{s}) \in \mathbb{R}^{n_\mathbf{q}\times n_\mathbf{q}}$, the generalised Coriolis-Centripetal matrix for the $n_\mathbf{q}$ DOF multibody system, given by:

$$\mathbf{C}^g_{RB}(\mathbf{q},\mathbf{s}) = \mathbf{P}\mathbf{M}^b_{RB}\dot{\mathbf{P}}^\top + \mathbf{P}\mathbf{C}^b_{RB}(\boldsymbol{\nu})\mathbf{P}^\top \tag{10}$$

where $\mathbf{C}^b_{RB} \in \mathbb{R}^{6N\times 6N} = diag\left(\mathbf{C}^{b_1}_{RB}(\boldsymbol{\nu})...\mathbf{C}^{b_N}_{RB}(\boldsymbol{\nu})\right)$.
On the right hand side of Equation 8, we have defined $\boldsymbol{\tau}^g_{RB} \in \mathbb{R}^{n_\mathbf{q}}$, given by:

$$\boldsymbol{\tau}^g_{RB} = \mathbf{P}\boldsymbol{\tau}^b_{RB} \tag{11}$$

where $\boldsymbol{\tau}^b_{RB} \in \mathbb{R}^{6N} = [\boldsymbol{\tau}^{b_1\top}_{RB}...\boldsymbol{\tau}^{b_N\top}_{RB}]^\top$ and $\boldsymbol{\tau}^{b_k}_{RB} = [\mathbf{v}^{b_k\top}_{c_k}, \boldsymbol{\omega}^{b_k\top}_{nb_k}]^\top$

### 2.3 External Forces

Fossen (2005) gives the external forces acting on a single surface vessel as:

$$\boldsymbol{\tau}^{b_k}_{RB} = -\mathbf{M}^{b_k}_A\dot{\boldsymbol{\nu}}_k - \boldsymbol{\mu}_k - \mathbf{G}^{b_k}\boldsymbol{\eta}^{b_k}_k + \boldsymbol{\tau}^{b_k}_E + \boldsymbol{\tau}^{b_k} \tag{12}$$

where $\mathbf{M}^{b_k}_A \in \mathbb{R}^{6\times 6}$ is the added inertia matrix, $\mathbf{G}^{b_k} \in \mathbb{R}^{6\times 6}$ is the hydrostatic restoring matrix, and $\boldsymbol{\tau}^{b_k} \in \mathbb{R}^6$ is the control force acting on body $k$. Further, the wave-excitation forces $\boldsymbol{\tau}^{b_k}_E \in \mathbb{R}^6$ are composed of:

$$\boldsymbol{\tau}^{b_k}_E = \boldsymbol{\tau}^{b_k}_{FK} + \boldsymbol{\tau}^{b_k}_D \tag{13}$$

where:

$\boldsymbol{\tau}^{b_k}_{FK}$ = generalised (Cartesian) Froude-Krylov forces
$\boldsymbol{\tau}^{b_k}_D$ = generalised (Cartesian) Diffraction forces

The $\boldsymbol{\mu}_k$ represents the convolution integral term in Cummins equation. Here a state-space approximation to the convolution integral is applied (Kristiansen and Egeland, 2003). The overall state space model for the single vessel is thus given by:

$$\dot{\boldsymbol{\eta}}_k = \mathbf{J}^{b_k}\boldsymbol{\nu}_k \tag{14}$$
$$\mathbf{M}^{b_k}\dot{\boldsymbol{\nu}}_k + \mathbf{C}^{b_k}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\mu}_k + \mathbf{G}^{b_k}\boldsymbol{\eta}^{b_k}_k = \boldsymbol{\tau}^{b_k}_E + \boldsymbol{\tau}^{b_k}$$
$$\dot{\boldsymbol{\chi}}_k = \mathbf{A_r}\boldsymbol{\chi}_k + \mathbf{B_r}\boldsymbol{\nu}_k \tag{15}$$
$$\boldsymbol{\mu}_k = \mathbf{C_r}\boldsymbol{\chi}_k + \mathbf{D_r}\boldsymbol{\nu}_k \tag{16}$$

where $\boldsymbol{\chi}_k(0) = \mathbf{0}$.
Utilising Equations 11, 6 and 7, it becomes possible to extend the state-space model (Equations 14:16) to $N$ interconnected surface vessels:

$$\dot{\mathbf{q}} = \mathbf{s} \tag{17}$$
$$\mathbf{M}^g\dot{\mathbf{s}} + \mathbf{C}^g_{RB}(\mathbf{q},\mathbf{s})\dot{\mathbf{s}} + \boldsymbol{\mu} + \mathbf{G}^b\mathbf{q} = \boldsymbol{\tau}^g_E + \boldsymbol{\tau}^g_c \tag{18}$$
$$\dot{\boldsymbol{\chi}}_k = \mathbf{A}^{\mathbf{b_k}}_\mathbf{r}\boldsymbol{\chi}_k + \mathbf{B}^{\mathbf{b_k}}_\mathbf{r}\boldsymbol{\nu}_k \tag{19}$$
$$\boldsymbol{\mu}^{b_k}_k = \mathbf{C}^{\mathbf{b_k}}_\mathbf{r}\boldsymbol{\chi}_k + \mathbf{D}^{\mathbf{b_k}}_\mathbf{r}\boldsymbol{\nu}_k \tag{20}$$

where $\mathbf{M}^b \in \mathbb{R}^{6N\times 6N} = \mathbf{M}^b_{RB} + \mathbf{M}^b_A$ and $\mathbf{M}^g \in \mathbb{R}^{n_\mathbf{q}\times n_\mathbf{q}}$ is given by:

$$\mathbf{M}^g = \mathbf{P}\mathbf{M}^b\mathbf{P}^\top \tag{21}$$

and $\mathbf{G}^g \in \mathbb{R}^{n_\mathbf{q}\times n_\mathbf{q}}$ is given by:

$$\mathbf{G}^g = \mathbf{P}\mathbf{G}^b\mathbf{P}^\top \tag{22}$$

where $\mathbf{G}^b \in \mathbb{R}^{6N\times 6N}$ is given by:

$$\mathbf{G}^b = diag(\mathbf{G}^{b_1}, ..., \mathbf{G}^{b_k}, ..., \mathbf{G}^{b_N}) \tag{23}$$

and $\mathbf{G}^{b_k} \in \mathbb{R}^{6\times 6}$ is given by:

$$\mathbf{G}^{b_k} = diag(0, 0, \rho g A_{WP}(0), \rho g\nabla\overline{GM}_T, \rho g\nabla\overline{GM}_L) \tag{24}$$

Finally, we note that

$$\boldsymbol{\mu}^g = \mathbf{P}\boldsymbol{\mu}^b \tag{25}$$

where: $\boldsymbol{\mu}^b \in \mathbb{R}^{6N} = [\boldsymbol{\mu}^{b_k\top}_k, ..., \boldsymbol{\mu}^{b_N\top}_N]^\top$,

### 2.4 Hydrodynamics

The hydrodynamic software package WAMIT was used to obtain added mass and radiation damping coefficients in the range $0.5 < \omega < 12$ rad/s (WAMIT, 2006). WAMIT outputs coupling coefficients for multibody systems. For example, the added mass matrix for a 2 body system is given by $\mathbf{A}(\omega) \in \mathbb{R}^{12\times 12}$:

With some transformations, detailed below, it becomes possible to incorporate the $6N \times 6N$ added mass and radiation damping matrices into the model developed in Section 3.1. This is a major advantage of the method. Similarly the radiation damping matrix for a given frequency is given by $\mathbf{B}(\omega) \in \mathbb{R}^{12\times 12}$. Although further processing is required following frame transformation, the resulting $6N$ force vector $\boldsymbol{\mu}_k$, expressed in $b_k$-frame, is easily incorporated using the velocity transformation matrix $\mathbf{P}$ (Equation 11).

All quantities output by WAMIT are expressed in a Cartesian data frame with origin on the free surface in the same vertical line as the center of gravity, as shown in Figure 2.4.
Hence it is necessary utilise the following transform, given in Fossen (2005):

$$\mathbf{T}^{h_k}_{data_k} = \begin{bmatrix} \mathbf{R}^{h_k}_{data_k} & 0 \\ 0 & \mathbf{R}^{h_k}_{data_k} \end{bmatrix} \tag{26}$$

where:

$$\mathbf{R}^{h_k}_{data_k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{27}$$

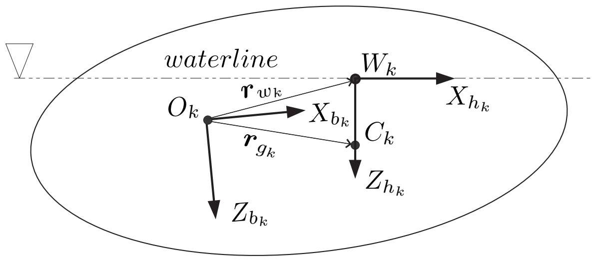


Fig. 1. Definitions of coordinate origins on body $k$: $W_k$ (waterline), $C_k$ (centre of gravity) and $O_k$ (equations of motion). The $h_k$-frame is located in $W_k$ and the $b_k$-frame is located in $O_k$

It now becomes necessary to transform the quantities above expressed in the $h_k$-frame into the $b_k$-frame. The $\mathbf{J}^{h_k}$ transform converts quantities from the coordinate frames used in hydrodynamic software to the body-fixed frames convenient for control modeling. We assume that the oscillations $\delta\boldsymbol{\Theta}_k$ of the $b_k$-frame about the $h_k$-frame are given by the tranform(Fossen and Smøgeli, 2004):

$$\mathbf{J}^{h_k}(\delta\boldsymbol{\Theta}_k) = \begin{bmatrix} \mathbf{R}^{h_k}_{b_k}(\delta\boldsymbol{\Theta}_k) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{R}^{h_k}_{b_k}(\delta\boldsymbol{\Theta}_k) \end{bmatrix} \mathbf{H}(\mathbf{r}^{b_k}_{w_k}) \tag{28}$$

where $\delta\boldsymbol{\Theta}_k$ signifies small angles, and $\mathbf{R}^{h_k}_{b_k}(\delta\boldsymbol{\Theta}_k) \in SO(3)$ is given by:

$$\mathbf{R}^{h_k}_{b_k}(\delta\boldsymbol{\Theta}_k) = \begin{bmatrix} 1 & -\delta\psi & \delta\theta \\ \delta\psi & 1 & -\delta\phi \\ -\delta\theta & \delta\phi & 1 \end{bmatrix} \tag{29}$$

and we have assumed $\dot{\mathbf{J}}^{h_k}(\delta\boldsymbol{\Theta}_k) \approx \mathbf{0}_{6\times6}$.

For many applications, the roll, pitch and yaw oscillations of the $b_k$-frame with respect to the $h_k$-frame, $\delta\phi_k, \delta\theta_k$ and $\delta\psi_k$, will be small, such that:

$$\mathbf{R}^{h_k}_{b_k}(\delta\boldsymbol{\Theta}_k) \approx \mathbf{I}_{3\times3} \tag{30}$$

and:

$$\mathbf{J}^{h_k}(\delta\boldsymbol{\Theta}_k) \overset{\delta\boldsymbol{\Theta}\ \text{small}}{\approx} \mathbf{H}(\mathbf{r}^{b_k}_{w_k}) \tag{31}$$

The necessary transform, fully derived in Fossen and Smøgeli (2004), may be stated here as:

$$\boldsymbol{\tau}^{h_k} = \mathbf{J}^{h_k}(\delta\boldsymbol{\Theta}_k)\boldsymbol{\tau}^{b_k} \tag{32}$$

## 3. EXAMPLE: TWO-BODY MARINE SYSTEM

Wave-tank experiments were carried out on the hinged-barge system shown in Figure 2. In the experiments, the system acted as a wave-energy converter, however this application is not considered here. A series of static, decay, regular and irregular wave tests were conducted on the device. Full details are documented in Ó'Catháin (2006).

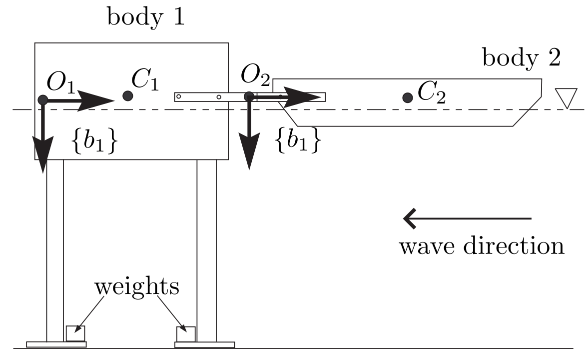


Fig. 2. Two-body marine system

### *3.1 Rigid Body Dynamics*

$$\mathbf{r}^{b_1}_{o_2} = [L_{o_2}, 0, 0]^\top \tag{33}$$

$$\mathbf{r}^{b_1}_{c_1} = [L_{c_1}, 0, 0]^\top \mathbf{r}^{b_2}_{c_2} = [L_{c_2}, 0, 0]^\top \tag{34}$$

$$\boldsymbol{\omega}^{b_1}_{nb_1} = [0, \theta_1, 0]^\top \quad \boldsymbol{\omega}^{b_2}_{nb_2} = [0, \theta_2, 0]^\top \tag{35}$$

$$\dot{\boldsymbol{\omega}}^{b_1}_{nb_1} = [0, \dot\theta_1, 0]^\top \quad \dot{\boldsymbol{\omega}}^{b_2}_{nb_2} = [0, \dot\theta_2, 0]^\top \tag{36}$$

$$\begin{aligned}
\mathbf{v}^{b_1}_{c_1} &= \mathbf{v}^{b_1}_{o_1} + \boldsymbol{\omega}^{b_1}_{nb_1} \times \mathbf{r}^{b_1}_{c_1} && (37)\\
&= \mathbf{S}(\boldsymbol{\omega}^{b_1}_{nb_1})\mathbf{r}^{b_1}_{c_1} && (38)\\
&= [0, 0, -L_{c_1}\dot\theta_1]^\top && (39)\\
\mathbf{v}^{b_2}_{c_2} &= \mathbf{v}^{b_2}_{o_2} + \boldsymbol{\omega}^{b_2}_{nb_2} \times \mathbf{r}^{b_2}_{c_2} && (40)\\
&= \mathbf{R}^{b_2}_{b_1}(\boldsymbol{\omega}^{b_1}_{nb_1} \times \mathbf{r}^{b_1}_{o_1}) + \boldsymbol{\omega}^{b_2}_{nb_2} \times \mathbf{r}^{b_2}_{c_2} && (41)\\
&= \mathbf{R}^{b_2}_{b_1}\mathbf{S}(\boldsymbol{\omega}^{b_1}_{nb_1})\mathbf{r}^{b_1}_{o_1} + \mathbf{S}(\boldsymbol{\omega}^{b_2}_{nb_2})\mathbf{r}^{b_2}_{c_2} && (42)\\
&= [L_1 s_{21}\dot\theta_1, 0, -L_1 c_{21}\dot\theta_1 - L_{c_1}\dot\theta_1]^\top && (43)
\end{aligned}$$

where where we have defined $c_{21} = \cos(\theta_2 - \theta_1)$, $s_{21} = \sin(\theta_2 - \theta_1)$, and the rotation matrix $R^{b_2}_{b_1}$ describing rotations, in the $x - z$ plane, of the $b_2$-frame about the $b_1$-frame is given by:

$$\begin{aligned}
\mathbf{R}^{b_2}_{b_1} &= \mathbf{R}^{b_1\,-1}_{b_2} = \mathbf{R}^{b_1\,\top}_{b_2} && (44)\\
&= \begin{bmatrix} c_{21} & 0 & -s_{21} \\ 0 & 1 & 0 \\ s_{21} & 0 & s_{21} \end{bmatrix} && (45)
\end{aligned}$$

Now we choose $\mathbf{q} = [\theta_1, \theta_2]^\top$. Hence $\mathbf{s} = [\dot\theta_1, \dot\theta_2]^\top$, and it follows that:

$$\frac{\partial \mathbf{v}^{b_1}_{c_1}}{\partial \mathbf{s}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -Lc_1 & 0 \end{bmatrix} \tag{46}$$

$$\frac{\partial \mathbf{v}^{b_2}_{c_2}}{\partial \mathbf{s}} = \begin{bmatrix} L_1 s_{21} & 0 \\ 0 & 0 \\ -L_1 c_{21} & -L_{c_2} \end{bmatrix} \tag{47}$$

Also:

$$\frac{\partial \boldsymbol{\omega}^{b_1}_{nb_1}}{\partial \mathbf{s}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial \boldsymbol{\omega}^{b_2}_{nb_2}}{\partial \mathbf{s}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{48}$$

This gives the *partial velocity matrix* $\mathbf{P} \in \mathbb{R}^{2\times 12}$ as:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & -Lc_1 & 0 & 1 & 0 & L_1 s_{21} & 0 & -L_1 c_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -L_{c_2} & 0 & 1 & 0 \end{bmatrix} \quad (49)$$

From Equations 9 and 10 the following expression for $\mathbf{M}^g_{RB} \in \mathbb{R}^{2\times 2}$ and $\mathbf{C}^g_{RB}(\mathbf{q}, \mathbf{s}) \in \mathbb{R}^{2\times 2}$ may be calculated:

$$\mathbf{M}^g_{RB} = \begin{bmatrix} I_{1y} + m_1 L_{c_1}^2 + m_2 L_1^2 & m_2 L_1 L_{c_2} c_{21} \\ m_2 L_1 L_{c_2} c_{21} & I_{2y} + m_2 L_{c_2}^2 \end{bmatrix} \quad (50)$$

$$\mathbf{C}^g_{RB}(\mathbf{q}, \mathbf{s}) = \begin{bmatrix} 0 & -m_2 L_1 L_{c_2} s_{21} \dot{\theta}_2 \\ m_2 L_1 L_{c_2} s_{21} \dot{\theta}_1 & 0 \end{bmatrix} \quad (51)$$

### 3.2 Hydrodynamics

Using the state-space approximation of Kristiansen and Egeland (2003) for the convolution term in the Cummins equation (Cummins, 1962), we find:

$$\mu_1^{b_1} = \sum_i \mu_{1,i} \qquad i = 1, 3, 5, 7, 9, 11 \quad (52)$$

$$\mu_3^{b_1} = \sum_i \mu_{3,i} \qquad i = 1, 3, 5, 7, 9, 11 \quad (53)$$

$$\mu_5^{b_1} = \sum_i \mu_{5,i} \qquad i = 1, 3, 5, 7, 9, 11 \quad (54)$$

$$\mu_7^{b_2} = \sum_i \mu_{7,i} \qquad i = 1, 3, 5, 7, 9, 11 \quad (55)$$

$$\mu_9^{b_2} = \sum_i \mu_{9,i} \qquad i = 1, 3, 5, 7, 9, 11 \quad (56)$$

$$\mu_{11}^{b_2} = \sum_i \mu_{11,i} \qquad i = 1, 3, 5, 7, 9, 11 \quad (57)$$

$$\boldsymbol{\mu}^b = \left[\mu_1^{b_1}, 0, \mu_3^{b_1}, 0, \mu_5^{b_1}, 0, \mu_7^{b_2}, 0, \mu_9^{b_2}, 0, \mu_{11}^{b_2}, 0\right]^\top \quad (58)$$

Hence $\boldsymbol{\mu}^g \in \mathbb{R}^{2\times 2}$ is given by:

$$\boldsymbol{\mu}^g = \mathbf{P}\boldsymbol{\mu}^b \quad (59)$$

Satisfactory agreement with the impulse response functions are achieved using an Order 6 state-space model of the convolution integral term.
$\mathbf{A}(\infty)$ is taken at the highest value of $\omega$ calculated, i.e. $\omega = 12$ rad/s. This is a somewhat crude approximation, however the weakness of the state-space approximation presented by Kristiansen and Egeland (2003) , is that good values for $\mathbf{A}(\infty)$ are difficult to obtain. This is due to the need for ever smaller panel sizes as frequency increases, and hence prohibitively long run-times. The method presented in Perez and Lande (2006) addresses this shortcoming.

Using the hydrostatic restoring matrix expression of 22 we find $\mathbf{G}^g \in \mathbb{R}^{2\times 2}$:

$$\mathbf{G}^g = \begin{bmatrix} L_{c_1}^2 A_{\mathrm{WP}_1}(0) + L_1^2 A_{\mathrm{WP}_2}(0) + \nabla_1 GM_{L_1} & L_1 L_{c_2} A_{\mathrm{WP}_2}(0) \\ L_1 L_{c_2} A_{\mathrm{WP}_2}(0) & L_{c_2}^2 A_{\mathrm{WP}_2}(0) + \nabla_2 GM_{L_2} \end{bmatrix} \quad (60)$$

### 3.3 Validation Against Experimental Results

In experiments, regular head waves excited the *two-body* system shown in Figure 2. Waves of height $H = 60mm$ over a range of 14 frequencies were applied, with the heave response of body two in *ned*-coordinates recorded over this interval. The response of the model derived in Sections 3.1 and 3.2 is simulated over the same frequency range. Figure 3 shows the response in heave of body two for a frequency of $\omega = 5$ rad/s.

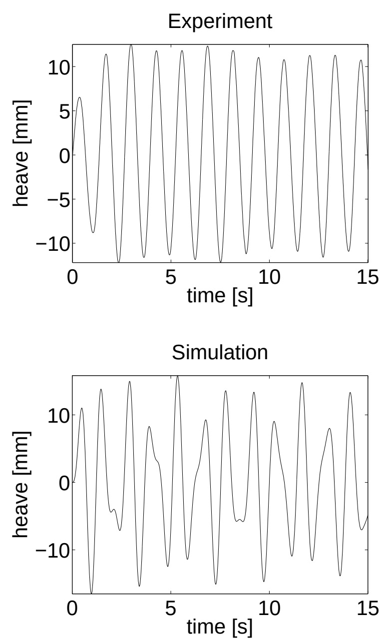


Fig. 3. Comparison of Body 2 heave response to regular wave excitation of height $H = 60$ mm and $\omega = 5$ rad/s

Figure 4 shows the response in heave of body two for a frequency of $\omega = 9$ rad/s. The nonlinear oscillations in the simulations are considerably stronger than the experiments at this frequency. However, the response at this higher frequency is less important than the response in the medium frequency range shown in Figure 3.

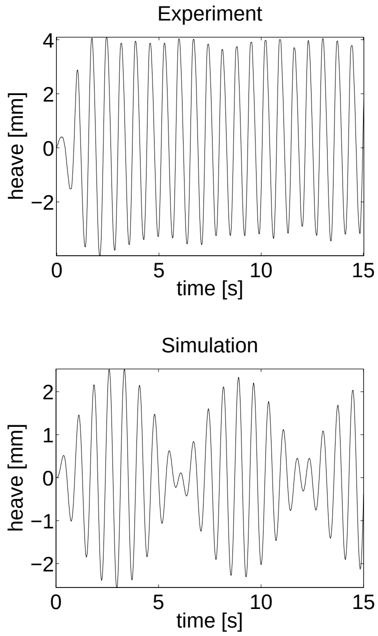


Fig. 4. Comparison of Body 2 heave response to regular wave excitation of height $H = 60$ mm and $\omega = 9$ rad/s

## CONCLUSIONS

A method for formulating the time-domain equations of motion for multibody marine systems has been derived. Equations of motion for a two-body interconnected marine structure have been presented, and reasonable agreement with experimental data at two practical frequencies has been demonstrated. The modeling method used is easily extendable to an arbitrary number of interconnected bodies. Since the method retains the vectorial formulation of Fossen (2002), it has potential for useful application in the field of marine control.

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## REFERENCES

Cummins, W.E. (1962). The impulse response function and ship motions. *Technical Report 1661. David Taylor Model BasinDTNSRDC.*

Egeland, O. and J.T. Gravdahl (2002). *Modeling and Simulation for Automatic Control*. Marine Cybernetics.

Fossen, Thor I. (2002). *Marine Control Systems: Guidance,Navigation, and Control of Ships, Rigs and Underwater Vehicles*. Marine Cybernetics.

Fossen, T.I. (2005). A nonlinear unified state-space model for ship maneuvering and control in a seaway. *Journal of Bifurication and Chaos*.

Fossen, T.I. and Ø. Smøgeli (2004). Nonlinear time-domain strip theory formulation for low-speed maneuvering and station-keeping. *Modelling,Identification and Control* **25**(4), 201–221.

Jordán, M.A. and R. Beltrán-Aguedo (2004). Optimal identification of potential-radiation hydrodynamics for moored floating structures-a new general approach in state space. *Ocean Engineering* **31**, 1859–1914.

Kristiansen, E. and O. Egeland (2003). Frequency-dependant added mass in models for controller design for wave motion damping. In: *Proc.6th MCMC*. Girona, Spain.

McCabe, Andrew P., Alan Bradshaw and Martin B. Widden (2005). A time-domain model of a floating body using transforms. In: *Proc.6th European Wave and Tidal Energy Conference(EWTEC)*. Glasgow, Scotland.

Ó'Catháin, M. (2006). Control-oriented modeling of a hinged barge wave-energy converter. Master's thesis. National University of Ireland (NUI), Maynooth. Dept. Electronic Engineering. To appear.

Perez, T. and O. Lande (2006). A frequency-domain approach to modeling and identification of the force to motion vessel response. In: *7th IFAC Conference on Manoeuvring and Control of Marine Vessels MCMC*. Portugal.

Shabana, A. (2005). *Dynamics of Multibody Systems*. 3rd ed.. Cambridge.

SNAME (1950). The society of naval architects and marine engineers. nomenclature for treating the motion of a submerged body through a fluid.. *Technical and Research Bulletin*.

WAMIT (2006). *WAMIT User-Manual*. WAMIT Inc. Massachusetts, USA. Version 6.3.

Yu, Z. and J. Falnes (1995). State-space modelling of a vertical cylinder in heave. *Applied Ocean Research* **17**(5), 265–275.