$\mathcal{H}_\infty$ Almost Synchronization for Non-Identical Introspective Multi-Agent Systems Under External Disturbances

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***Abstract*— In this paper, synchronization for multi-agent systems subject to external disturbances is studied, and the notion of "$\mathcal{H}_\infty$ almost synchronization" is introduced. The objective is to suppress the impact of disturbances on the synchronization error dynamics in terms of the $\mathcal{H}_\infty$ norm of the corresponding closed-loop transfer function to any arbitrarily value. We focus on networks of non-identical linear, right-invertible agents under directional communication links. The key assumption is that agents are introspective; that is, they have a partial knowledge about their own states.**

I. INTRODUCTION

Dynamical networked systems have received a great deal of attention in the past decade. The application is widespread and spans formations of autonomous space, land and marine vehicles, sensor networks, coordinated decision making, to name a few. A thorough coverage of earlier work can be found in [1]–[3] and references therein.

Many results consider consensus of first order or second order integrator dynamics; see e.g. [4], [5]. They basically rely on full state information from the network and design static decentralized protocols.

Consensus in general linear agents is addressed in [6]–[10] where partial-state information is given to each agent via the network and dynamic protocols are introduced. Reference [7] proposes a low-gain approach by filtering the information that each agent receives whereas [6] considers self-feedback for all agents. A significant breakthrough in the design of dynamic protocols is presented in [8] where conventional observers are expanded to distributed observers while agents are capable of exchanging information about their own estimates over the network. The result is extended to LQR-based optimal design in [9]. More general networks and regulation of output consensus are studied in [10].

Networks of non-identical agents have not yet been investigated thoroughly. The common assumption is that agents are introspective; that is, agents possess some knowledge about their own states. For networks of nonlinear agents, [11] presents criteria for state consensus. Output consensus for weakly minimum-phase systems of relative degree one is studied in [12] where local feedbacks are utilized to decouple zero dynamics and create a single integrator system. Embedding additional models within agents, [13] proposes a controller for SISO minimum-phase systems. A method to represent a network of non-identical agents as a network of asymptotically identical agents is developed in [14], and a decentralized protocol is designed to reach output consensus. Relaxing the self-knowledge (introspective) assumption, [15] puts forward a distributed dynamic protocol for networks of linear, non-identical and non-introspective agents.

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A. The Topic of The Paper

The paper brings forth the notion of "$\mathcal{H}_\infty$ almost synchronization". In fact, we study the output synchronization problem for a network of linear, right invertible agents with non-identical dynamics of any order, subject to external disturbances and under directed interconnection topologies. We aim to construct a family of parameterized linear time-invariant protocols based on a distributed observer such that i) synchronization is accomplished in the absence of disturbance, and ii) disturbance is attenuated in the controlled output (which is selected as a function of the disagreement between any pair of agents) to any arbitrarily small value in the sense of $\mathcal{H}_\infty$ norm of the transfer function. Thus, any desired accuracy of synchronization may be achieved.

Previous Work: Synchronization in the presence of external disturbances has been the topic of fairly few papers which usually focus on networks of identical agents. For bidirectional exchange links, [8] shows that the distributed $\mathcal{H}_\infty$ optimization problem is converted to the $\mathcal{H}_\infty$ optimization problem for a set of independent agents if the goal is to reduce the $\mathcal{H}_\infty$ norm of the transfer function from disturbance to the output of each agent.

For directed networks of first-order and second-order integrator systems under external disturbances, [16] and [17] develop static LMI-based $\mathcal{H}_\infty$ controllers. For identical agents, [18] seeks a distributed observer with $\mathcal{H}_\infty$ robust performance using the concept of vector dissipativity.

However, we intend to deal with networks of non-identical agents so as to propose a method to achieve (almost) synchronization with *any* arbitrarily small $\mathcal{H}_\infty$ gain. Note that this problem cannot be broken down into $\mathcal{H}_\infty$ control problem of N independent systems as proposed by [8].

B. Notations and Preliminaries

Throughout the paper, I_n denotes the identity matrix of dimension n, and $\mathbf{1}_n \in \mathbb{R}^n$ means a vector whose entries are all one. Matrix A is denoted $A = [a_{ij}]$ where a_{ij} represents the element (i,j) of A. Given a matrix A, A^{T}, A^{H} and $\|A\|$ indicate transpose, complex conjugate transpose, and induced 2-norm of A. The Kronecker product of A and B is symbolized by $A \otimes B$. A block-diagonal matrix constructed by A_i's is denoted $\mathrm{diag}\{A_i\}$ for $i = 1, \cdots, n$. Also, $x = \mathrm{col}\{x_i\}$ for $i = 1, \cdots, n$ is adopted to denote $x = [x_1^{\mathrm{T}}, \cdots, x_n^{\mathrm{T}}]^{\mathrm{T}}$ where x_i's are vectors. The open left-half and right-half complex planes are represented by $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. The real part of a complex number λ, is represented by $\mathrm{Re}\{\lambda\}$. For transfer function $T(s)$, the $\mathcal{H}_\infty$ norm is denoted $\|T(s)\|_\infty$.

Let $\mathcal{L}$ be a weighted directed graph with n nodes. If there is an edge from node j to node i, $a_{ij} > 0, a_{ij} \in \mathbb{R}$ is assigned to the edge; otherwise, $a_{ij} = 0$. If the graph is not allowed to have self-loops, $a_{ii} = 0$. $A_{\mathcal{L}} = [a_{ij}]$ is the weighted adjacency matrix of $\mathcal{L}$. The Laplacian of $\mathcal{L}$ is denoted by $L = [l_{ij}]$ where $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. It implies that $\mathbf{1}_n$ is a right eigenvector of L associated with the eigenvalue at zero. A digraph is said to have a directed spanning tree if there is a node from which a directed path exists to every other nodes. L has a simple eigenvalue at zero and all the other eigenvalues are in $\mathbb{C}^+$, provided that $\mathcal{L}$ contains a directed spanning tree [5].

II. Multi-Agent Systems

A multi-agent system is referred to a network of multiple-input multiple-output agents described by LTI models as

$$\text{Agent } i : \begin{cases} \dot{\bar{x}}_i = A_i \bar{x}_i + B_i \bar{u}_i + G_i \bar{w}_i & \text{(1a)} \\ y_i = C_i \bar{x}_i & \text{(1b)} \end{cases}$$

in which $i \in \mathbb{S} \triangleq \{1, \cdots, N\}$, $\bar{x}_i \in \mathbb{R}^{n_i}$, $\bar{u}_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^p$, $\bar{w}_i \in \mathbb{R}^{\omega_i} : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \bar{w}_i^{\mathrm{T}} \bar{w}_i \, \mathrm{d}t < \infty$ are the agent's state, input, output, and external disturbance, respectively. Agents are introspective; that is, each agent possesses partial knowledge of its own states through the local measurement:

$$y_{m,i} = C_{m,i}\, \bar{x}_i \tag{1c}$$

Furthermore, agents are allowed to exchange information according to the network's communication topology which is characterized by a directed graph $\mathcal{L}$, with no self loops, associated with the adjacency matrix $A_{\mathcal{L}} = [a_{ij}]$ and the Laplacian matrix $L = [l_{ij}]$. Accordingly, each agent has access to a weighted linear combination of outputs. In particular, the network measurement for agent $i \in \mathbb{S}$ is:

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) \quad \Rightarrow \quad \zeta_i = \sum_{j=1}^{N} l_{ij} y_j \tag{1d}$$

Also, the agents can exchange additional information over the network. The transmission of the information conforms with the communication topology and facilitates the design of a distributed observer for the network. Thus, agent i has access to the following quantity:

$$\hat{\zeta}_i = \sum_{j=1}^{N} l_{ij} \eta_j \tag{1e}$$

where $\eta_j \in \mathbb{R}^p$ depends on the protocol's states of agent j, and will be specified later in Section III-A

Assumption 1: We make the following assumptions for agent $i \in \mathbb{S}$.

1) (A_i, B_i, C_i) is right-invertible;
2) (A_i, B_i) is stabilizable and (A_i, C_i) is detectable;
3) $(A_i, C_{m,i})$ is detectable;

III. $\mathcal{H}_\infty$ Almost Synchronization

This section introduces the notion of "$\mathcal{H}_\infty$ Almost Synchronization", and proposes a protocol to achieve it.

Problem Formulation We define the following vectors which are formed by stacking the corresponding vectors:

$$\boldsymbol{w} \triangleq \mathrm{col}\{\bar{w}_i\}, \qquad \boldsymbol{u} \triangleq \mathrm{col}\{\bar{u}_i\}, \qquad \boldsymbol{\zeta} \triangleq \mathrm{col}\{\zeta_i\}$$

for $i \in \mathbb{S}$. Let the mutual disagreement between any pair of agents be defined as: $\mathfrak{e}_{i,j} \triangleq y_i - y_j \; \forall i, j \in \mathbb{S}, i > j$. The stacking column vector of all mutual disagreements is denoted $\mathfrak{e}$. We define the following transfer function with the appropriate dimension: $\mathfrak{e} = T_{w\mathfrak{e}}(s)\boldsymbol{w}$.

Definition 1: Consider the multi-agent system (1) with a communication topology $\mathcal{L}$. Given a set of network graphs $\mathcal{G}$ and an arbitrary $\gamma > 0$, the "$\mathcal{H}_\infty$ almost synchronization" problem is to find, if possible, a linear time-invariant dynamic protocol such that, for any $\mathcal{L} \in \mathcal{G}$, the closed-loop transfer function from $\boldsymbol{w}$ to $\mathfrak{e}$ satisfies $\|T_{w\mathfrak{e}}(s)\|_\infty < \gamma$. ◀

To be able to provide a solution, we define a set of network graphs as bellow.

Definition 2: For given $\beta > 0$ and integer $N_0 \geq 1$, $\mathcal{G}_\beta$ is the set of graphs composed of N nodes where $N \leq N_0$ such that every $\mathcal{L} \in \mathcal{G}_\beta$ has a directed spanning tree, and every nonzero eigenvalue of its Laplacian, denoted λ_i, $i = 1, \cdots, N$, satisfies $\mathrm{Re}\{\lambda_i\} > \beta$. ◀

Theorem 1: Under Assumption 1 and for the set $\mathcal{G}_\beta$, the problem of $\mathcal{H}_\infty$ almost synchronization is solvable; specifically, there exists a family of linear time-invariant dynamic protocols, parameterized in terms of a tuning parameter $\epsilon \in (0, 1]$, of the form

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\epsilon)\chi_i + \mathcal{B}_i(\epsilon)\, \mathrm{col}\{\zeta_i, \hat{\zeta}_i, y_{m,i}\} & \text{(2a)} \\ \bar{u}_i = \mathcal{C}_i(\epsilon)\chi_i + \mathcal{D}_i(\epsilon)\, \mathrm{col}\{\zeta_i, \hat{\zeta}_i, y_{m,i}\} & \text{(2b)} \end{cases}$$

where $\chi_i \in \mathbb{R}^{q_i}$ and $i \in \mathbb{S}$ such that

(i) given $\beta > 0$, there exists an $\epsilon_1^* \in (0, 1]$ such that for every $\epsilon \in (0, \epsilon_1^*]$, synchronization is achieved in the absence of disturbance; i.e. $\forall \epsilon \in (0, \epsilon_1^*]$ when $\boldsymbol{w} = 0$, $\mathfrak{e}_{i,j} = y_i - y_j \to 0$, $\forall i, j \in \mathbb{S}$, $i > j$, as $t \to \infty$.

(ii) given $\gamma > 0$, there exists an $\epsilon_2^* \in (0, \epsilon_1^*]$ such that for every $\epsilon \in (0, \epsilon_2^*]$, the closed-loop transfer function from $\boldsymbol{w}$ to $\mathfrak{e}$ satisfies $\|T_{w\mathfrak{e}}(s)\|_\infty < \gamma$. □

The proof of Theorem 1 is presented in the subsequent sections in a constructive way. It is shown that Theorem 1 follows directly from Lemmas 1 and 2.

In the following, first, we present the solution for a subclass of multi-agent systems (1); then, we explain how to generalize the method.

A. Special Case

In this section, a special class of multi-agent system (1) is taken into consideration, and it is demonstrated how we achieve $\mathcal{H}_\infty$ almost synchronization.

Consider a special class of multi-agent system (1) as:

$$\dot{x}_i = Ax_i + B(Mu_i + Rx_i) + E_i w_i \tag{3}$$
$$y_i = Cx_i \tag{4}$$

where $x_i \in \mathbb{R}^{pn_q}$, u_i, $y_i \in \mathbb{R}^p$, $w_i \in \mathbb{R}^{\omega_i}$, $E_i \in \mathbb{R}^{pn_q \times \omega_i}$,

$$A = \begin{bmatrix} 0 & \mathrm{I}_{p(n_q-1)} \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \mathrm{I}_p \end{bmatrix},\ C = \begin{bmatrix} \mathrm{I}_p & 0 \end{bmatrix} \tag{5}$$

M is nonsingular and R is a $p \times pn_q$ matrix. $n_q \geq 1$ is an integer. If $n_q = 1$, $A = 0$, $B = C = \mathrm{I}_p$. Each agent collects network measurements (1d) and (1e). This special class of multi-agent systems does not require local measurements (1c). Let $\epsilon \in (0, 1]$ be the tuning parameter, and define the scaling matrix S as

$$S = \mathrm{diag}\{\mathrm{I}_p, \epsilon\, \mathrm{I}_p, \cdots, \epsilon^{n_q-1}\mathrm{I}_p\} \in \mathbb{R}^{pn_q \times pn_q} \tag{6}$$

For $i \in \mathbb{S}$, construct the observer-based protocol of the form

$$\dot{\hat{x}}_i = A\,\hat{x}_i + B(Mu_i + R\hat{x}_i) - \epsilon^{-1}K(\zeta_i - \hat{\zeta}_i) \tag{7a}$$
$$u_i = \epsilon^{-n_q}M^{-1}FS\hat{x}_i \tag{7b}$$

where $\hat{x}_i \in \mathbb{R}^{pn_q}$; F and K are the gains to be specified. $\hat{\zeta}_i$ is given by (1e) where $\eta_i = C\hat{x}_i$. Consider the following structure for the observer gain:

$$K = \begin{bmatrix} K_1 \\ \bar{K}K_1 \end{bmatrix} \quad \text{where} \quad \begin{matrix} K_1 \in \mathbb{R}^{p\times p} \\ \bar{K} \in \mathbb{R}^{p(n_q-1)\times p} \end{matrix} \tag{8}$$

Before presenting the design procedure, we need to partition the system matrices as

$$A = \begin{bmatrix} 0_{p\times p} & C_1 \\ 0_{p(n_q-1)\times p} & A_1 \end{bmatrix},\ B = \begin{bmatrix} 0_{p\times p} \\ B_1 \end{bmatrix},\ E_i = \begin{bmatrix} E_{1,i} \\ E_{2,i} \end{bmatrix}$$

where $C_1 = [\mathrm{I}_p, 0, \cdots, 0] \in \mathbb{R}^{p\times p(n_q-1)}$ and $E_{1,i} \in \mathbb{R}^{p\times \omega_i}$. Also partition $R = [R_1,\ \bar{R}]$ where $R_1 \in \mathbb{R}^{p\times p}$.

Design Procedure The gains of (7) are chosen as follows:

- Considering the controllable pair (A, B), choose F such that $A + BF$ is Hurwitz stable.
- Considering the observable pair $(A_1 + B_1\bar{R}, C_1)$, choose $\bar{K}$ such that $\widetilde{A}_z \triangleq A_1 + B_1\bar{R} - \bar{K}C_1$ is Hurwitz stable.
- Let $K_1 = K_1^{\mathrm{T}} < 0$.

Lemma 1: For any given $\gamma > 0$ and $\mathcal{G}_\beta$, there exists a sufficiently small $0 < \epsilon \leq 1$ such that the dynamic protocol (7) solves the $\mathcal{H}_\infty$ almost synchronization problem for multi-agent systems of the form (3) with any communication topology $\mathcal{L} \in \mathcal{G}_\beta$.

Proof: See Appendix I. ■

The proposed design method solves the problem using a parameterized protocol which achieves synchronization with any arbitrarily accuracy in terms of the $\mathcal{H}_\infty$ norm of the transfer function while the order of the protocol is fixed.

It is worth noting that having non-identical disturbance matrices in (3) significantly complicates $\mathcal{H}_\infty$ analysis for the network since it does not allow us to use input-output transformation as proposed by [19]. Thus, their result no longer holds.

B. Almost Identical Representation for Networks of Non-Identical Agents

In this section, we present a method to shape a multi-agent system of the form (1) into the form (3). The requisite for shaping is the local measurement (1c). Therefore, the result given in the preceding section is expanded to the general form of non-identical agents. Lemma 2 states the result formally.

Definition 3: Let $n_{q0} \geq 1$ be the maximum order of the infinite zeros of all triples (A_i, B_i, C_i), $i \in \mathbb{S}$. ◀

Lemma 2: Consider the multi-agent system (1) satisfying Assumption 1-(3). Let $n_q \geq n_{q0}$. For each agent, there exists a local dynamic compensator

$$\begin{cases} \dot{\bar{\tilde{x}}}_i = \widetilde{A}_i\bar{\tilde{x}}_i + \widetilde{B}_{1i}u_i + \widetilde{B}_{2i}y_{m,i} & \text{(9a)} \\ \bar{u}_i = \widetilde{C}_i\bar{\tilde{x}}_i + \widetilde{D}_{1i}u_i + \widetilde{D}_{2i}y_{m,i} & \text{(9b)} \end{cases}$$

such that the application of (9) to (1) can be written as

$$\begin{cases} \dot{x}_i = Ax_i + B(Mu_i + Rx_i) + E_{d,i}\bar{w}_i + \rho_i & \text{(10a)} \\ y_i = Cx_i & \text{(10b)} \end{cases}$$

where $R \in \mathbb{R}^{p\times pn_q}$ and nonsingular $M \in \mathbb{R}^{p\times p}$ are selected arbitrarily while A, B and C are as (5). Also, $\rho_i \in \mathbb{R}^p$ is evolved from the system described by

$$\begin{cases} \dot{\tilde{x}}_i = H_i\tilde{x}_i + E_{o,i}\bar{w}_i & \text{(11a)} \\ \rho_i = W_i\tilde{x}_i & \text{(11b)} \end{cases}$$

where H_i is Hurwitz stable.

Proof: See [20]. ■

Lemma 2 shows that a network of non-identical agents under disturbances can be converted to an almost identical network. As H_i is Hurwitz stable, $\tilde{x}_i$ and ρ_i have the same nature as $\bar{w}_i$; thus, one can redefine external disturbances as $w_i \triangleq \mathrm{col}\{\bar{w}_i, \tilde{x}_i\}$. Hence, the model (10) is recast as (3) where $E_i = [E_{d,i},\ W_i]$. Redefining disturbance changes the $\mathcal{H}_\infty$ norm; however, as the $\mathcal{H}_\infty$ norm of (11) is constant with respect to ϵ, it does not affect the solvability of the problem and $\mathcal{H}_\infty$ almost synchronization is solvable for any given $\mathcal{G}_\beta$ and $\gamma > 0$ by an appropriate choice of ϵ.

Remark 1: In Lemma 2, R and M can be chosen arbitrarily to satisfy the designer's wish. Actually, Lemma 2 states that agents can be shaped into the dynamics of any invertible system with no invariant zeros and of uniform rank n_q.

IV. Illustrative Example

The result is illustrated for a network consisting of $N = 4$ right-invertible agents with $p = 1$. The interconnection topology of the network is given by the digraph displayed in Fig. 1. The models of agents can be found in [20]. Disturbances are chosen $\bar{w}_1 = \sin(t)$, $\bar{w}_2 = 1$, $\bar{w}_4 = \sin(2t)$, and $\|\bar{w}_3\| \leq 5$ which is a random number normally distributed. The order of the infinite zeros of agent 1 to 4 are respectively 3, 2, 1, and 2. It is obtained that $n_{q0} = 3$.

Network Homogenization: The first step is to design a local output feedback for each agent to homogenize the multi-agent system. Interested reader is referred to [20] for detailed procedure and information. We choose $n_q = n_{q0}$. The resulting systems (after pre-compensation) have the form of (3) with $p = 1$ and $n_q = 3$. After pre-compensation, agent 1 to 4 are of orders 3, 3, 8 and 7, respectively. We choose $M = 1$ and $R = [0,\ -1,\ 0]$ to have $\lambda(A + BR) = \{0, \pm i\}$.

$\mathcal{H}_\infty$ *Almost Synchronization*: The appropriately shaped agents are now considered and the control law is generated according to (7). We select $K_1 = -1$. F and $\bar{K}$ are selected such that $\lambda(A + BF) = \{-3, -4, -5\}$ and $\lambda(\widetilde{A}_z) = \{-2, -3\}$. Fig. 2 shows the result for $\epsilon = 0.01$ and $\epsilon = 0.05$. The smaller ϵ is, the smaller ζ_i is, the more $\boldsymbol{w}$ is rejected from mutual disagreements; thus, more accurate synchronization can be obtained.

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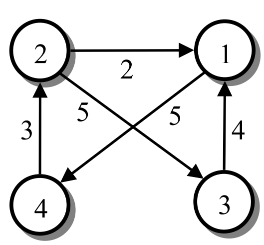

Fig. 1: Primary Network

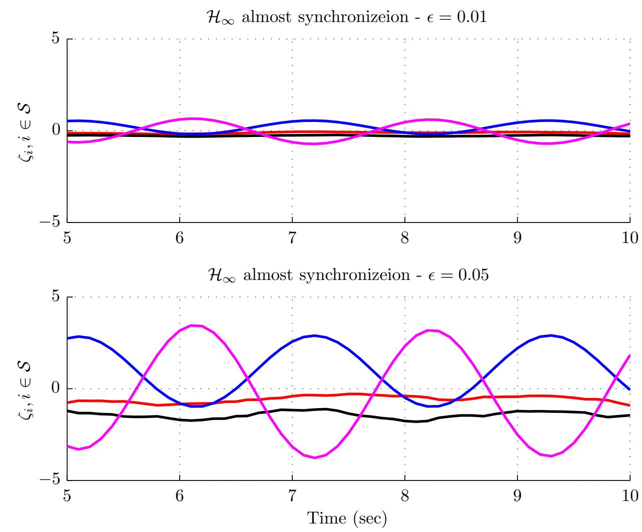

Fig. 2: $\mathcal{H}_\infty$ almost output synchronization. A blow-up of the simulation results.

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Appendix I
Proof of Lemma 1

Before embarking on the proof, we stress that the problem of $\mathcal{H}_\infty$ almost synchronization is solved by showing that the proposed protocol approximately decouples $\boldsymbol{\zeta}$ from $\boldsymbol{w}$ in the sense of $\mathcal{H}_\infty$ norm of the closed-loop transfer function from $\boldsymbol{w}$ to $\boldsymbol{\zeta}$. We shall show in Lemma 3 that a similar decoupling effect is seen between every $\mathfrak{e}_{i,j}$ and $\boldsymbol{w}$.

Closed-loop Equations: Let $x_i^* \triangleq x_i - \hat{x}_i$ be the observation error for agent i; then, in view of (3) and (7), the closed-loop equations for agent i can be written as

$$\dot{x}_i = Ax_i + BRx_i + \epsilon^{-n_q} BFS(x_i - x_i^*) + E_i w_i$$
$$\dot{x}_i^* = Ax_i^* + BRx_i^* + \epsilon^{-1} KC \sum_{j=1}^{N} l_{ij} x_j^* + E_i w_i$$

Let $R = [R_1,\ \bar{R}]$ with $\bar{R} = [R_2,\ \cdots,\ R_{n_q}]$ where $R_1, \cdots, R_{n_q} \in \mathbb{R}^{p\times p}$. Consider the following state transformations

$$e_i = Sx_i\,, \quad z_i = \bar{S}x_i^*\,, \quad \bar{S} \triangleq \begin{bmatrix} \mathrm{I}_p & 0 \\ -\epsilon\bar{K} & \epsilon\,\mathrm{I}_{p(n_q-1)} \end{bmatrix}$$

The closed-loop equations are then recast as:

$$\dot{e}_i = \epsilon^{-1}(A + BF)e_i + R_e e_i - \epsilon^{-1} BFS\bar{S}^{-1} z_i + SE_i w_i$$
$$\dot{z}_i = A_z z_i + R_z z_i + \epsilon^{-1} C^{\mathrm{T}} K_1 C \sum_{j=1}^{N} l_{ij} z_j + \bar{S}E_i w_i$$

where $R_e = B[\epsilon^{n_q-1}R_1,\ \epsilon^{n_q-2}R_2,\ \cdots,\ \epsilon R_{n_q-1},\ R_{n_q}]$ and

$$\begin{array}{l} R_z = \epsilon B\left[R_1 + \bar{R}\bar{K},\ \epsilon^{-1}\bar{R}\right] \\ \bar{K}' \triangleq A_1 - \bar{K}C_1 \end{array}, \quad A_z = \begin{bmatrix} C_1\bar{K} & \epsilon^{-1}C_1 \\ \epsilon\bar{K}'\bar{K} & \bar{K}' \end{bmatrix}$$

We split z_i into $z_{1,i} = Cz_i \in \mathbb{R}^p$ and $z_{2,i}$ such that $z_i = \mathrm{col}\{z_{1,i}, z_{2,i}\}$. Partition

$$F = \begin{bmatrix} F_1 & F_2 & \cdots & F_{n_q} \end{bmatrix}$$
$$\bar{K} = \begin{bmatrix} \bar{K}_1^{\mathrm{T}} & \bar{K}_2^{\mathrm{T}} & \cdots & \bar{K}_{n_q-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

where $F_j,\ \bar{K}_j \in \mathbb{R}^{p\times p}$ for $j = 1, \cdots, n_q$. Let $FS\bar{S}^{-1} = [F_1^*,\ F_2^*]$ where

$$F_1^* = F_1 + \sum_{s=1}^{n_q-1} \epsilon^s F_{s+1}\bar{K}_s$$
$$F_2^* = [F_2,\ \epsilon F_3,\ \cdots,\ \epsilon^{n_q-3}F_{n_q-1},\ \epsilon^{n_q-2}F_{n_q}]$$

Then, one may show the closed-loop equations as

$$\epsilon\dot{e}_i = (A + BF)e_i + \epsilon R_e e_i - BF_1^* z_{1,i} - BF_2^* z_{2,i} + \epsilon SE_i w_i$$
$$\epsilon\dot{z}_{1,i} = \epsilon C_1\bar{K}z_{1,i} + C_1 z_{2,i} + K_1 \sum_{j=1}^{N} l_{ij} z_{1,j} + \epsilon E_{1,i} w_i$$
$$\dot{z}_{2,i} = \epsilon\widetilde{E}_z z_{1,i} + \widetilde{A}_z z_{2,i} + \epsilon\widetilde{E}_{2,i} w_i$$

where $\widetilde{E}_{2,i} \triangleq E_{2,i} - \bar{K}E_{1,i}$ and $\widetilde{E}_z \triangleq \widetilde{A}_z\bar{K} + B_1R_1$. For $i \in \mathbb{S}$, consider the following notations

$$\mathbb{G} = \mathrm{diag}\{SE_i\}, \quad \widetilde{\mathbb{G}}_1 = \mathrm{diag}\{E_{1,i}\}, \quad \widetilde{\mathbb{G}}_2 = \mathrm{diag}\{\widetilde{E}_{2,i}\}$$
$$e = \mathrm{col}\{e_i\}, \qquad z_1 = \mathrm{col}\{z_{1,i}\}, \qquad z_2 = \mathrm{col}\{z_{2,i}\}$$

Considering the network Laplacian $L = [l_{ij}]$ for $i, j \in \mathbb{S}$, the closed-loop equations for the network are given by

$$\begin{aligned} \epsilon\dot{e} &= (\mathrm{I}_N \otimes (A + BF))\,e + (\mathrm{I}_N \otimes \epsilon R_e)e \\ &\quad - (\mathrm{I}_N \otimes BF_1^*)z_1 - (\mathrm{I}_N \otimes BF_2^*)z_2 + \epsilon\mathbb{G}\boldsymbol{w} \\ \epsilon\dot{z}_1 &= (\mathrm{I}_N \otimes \epsilon C_1\bar{K} + L \otimes K_1)z_1 + (\mathrm{I}_N \otimes C_1)z_2 + \epsilon\widetilde{\mathbb{G}}_1\boldsymbol{w} \\ \dot{z}_2 &= (\mathrm{I}_N \otimes \epsilon\widetilde{E}_z)z_1 + (\mathrm{I}_N \otimes \widetilde{A}_z)z_2 + \epsilon\widetilde{\mathbb{G}}_2\boldsymbol{w} \\ \boldsymbol{\zeta} &= (L \otimes C)e \end{aligned}$$

Reduced-order Dynamics: Let $\mathbf{1} \in \mathbb{R}^N$ be the right eigenvector of L for the eigenvalue at zero. Let $\mathbf{1}_L$ be the associated left eigenvector. Suppose the Jordan form of L is obtained using the matrix U as

$$U = \begin{bmatrix} \bar{U} & \mathbf{1} \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} \bar{U}_L^{\mathrm{T}} \\ \mathbf{1}_L^{\mathrm{T}} \end{bmatrix}$$

It implies that $\bar{U}_L^{\mathrm{T}}\bar{U} = \mathrm{I}_{N-1}$, $\bar{U}_L^{\mathrm{T}}\mathbf{1} = \mathbf{1}_L^{\mathrm{T}}\bar{U} = 0$, and $\mathbf{1}_L^{\mathrm{T}}\mathbf{1} = 1$. Thus, one can find

$$U^{-1}LU = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \qquad LU = \begin{bmatrix} \check{L} & 0 \end{bmatrix} \tag{14}$$

Thus, $LU = [\check{L},\ 0]$ where $\check{L} = \bar{U}\Delta$. Considering the transformation matrices $T_1 = (U^{-1} \otimes \mathrm{I}_{pn_q})$, $T_2 = (U^{-1} \otimes \mathrm{I}_p)$, and $T_3 = (U^{-1} \otimes \mathrm{I}_{\bar{p}})$ where $\bar{p} = p(n_q - 1)$, the following state transformations are introduced:

$$\begin{bmatrix} \bar{e} \\ e^* \end{bmatrix} = T_1 e\,, \quad \begin{bmatrix} \bar{z}_1 \\ z_1^* \end{bmatrix} = T_2 z_1\,, \quad \begin{bmatrix} \bar{z}_2 \\ z_2^* \end{bmatrix} = T_3 z_2$$

Denoting $\bar{N} = N - 1$, the states $\bar{e}$, $\bar{z}_1$ and $\bar{z}_2$ are vectors of the dimensions $\bar{N}pn_q$, $\bar{N}p$ and $\bar{N}\bar{p}$, respectively. Obviously, the states e^*, z_1^* and z_2^* are vectors of the dimensions pn_q, p and $\bar{p}$, respectively. Consider the following notations:

$$\begin{aligned} \widehat{\mathbb{G}}_e &= (\bar{U}_L^{\mathrm{T}} \otimes \mathrm{I}_{pn_q})\mathbb{G}, & \mathbb{G}_e^* &= (\mathbf{1}_L^{\mathrm{T}} \otimes \mathrm{I}_{pn_q})\mathbb{G} \\ \widehat{\mathbb{G}}_{z1} &= (\bar{U}_L^{\mathrm{T}} \otimes \mathrm{I}_p)\widetilde{\mathbb{G}}_1, & \mathbb{G}_{z1}^* &= (\mathbf{1}_L^{\mathrm{T}} \otimes \mathrm{I}_{pn_q})\widetilde{\mathbb{G}}_1 \\ \widehat{\mathbb{G}}_{z2} &= (\bar{U}_L^{\mathrm{T}} \otimes \mathrm{I}_{\bar{p}})\widetilde{\mathbb{G}}_2, & \mathbb{G}_{z2}^* &= (\mathbf{1}_L^{\mathrm{T}} \otimes \mathrm{I}_{\bar{p}})\widetilde{\mathbb{G}}_2 \end{aligned}$$

Therefore, the system dynamics are divided into two subsystems. One subsystem is of order $2pn_q\bar{N}$, and is given by

$$\begin{aligned} \epsilon\dot{\bar{e}} &= (\mathrm{I}_{\bar{N}} \otimes (A + BF))\bar{e} + (\mathrm{I}_{\bar{N}} \otimes \epsilon R_e)\bar{e} \\ &\quad - (\mathrm{I}_{\bar{N}} \otimes BF_1^*)\bar{z}_1 - (\mathrm{I}_{\bar{N}} \otimes BF_2^*)\bar{z}_2 + \epsilon\,\widehat{\mathbb{G}}_e\boldsymbol{w} \end{aligned} \tag{15a}$$

$$\begin{aligned} \epsilon\dot{\bar{z}}_1 &= (\mathrm{I}_{\bar{N}} \otimes \epsilon C_1\bar{K} + \Delta \otimes K_1)\bar{z}_1 \\ &\quad + (\mathrm{I}_{\bar{N}} \otimes C_1)\bar{z}_2 + \epsilon\,\widehat{\mathbb{G}}_{z1}\boldsymbol{w} \end{aligned} \tag{15b}$$

$$\dot{\bar{z}}_2 = (\mathrm{I}_{\bar{N}} \otimes \epsilon\widetilde{E}_z)\bar{z}_1 + (\mathrm{I}_{\bar{N}} \otimes \widetilde{A}_z)\bar{z}_2 + \epsilon\,\widehat{\mathbb{G}}_{z2}\boldsymbol{w} \tag{15c}$$

Note that $\boldsymbol{\zeta} = (\check{L} \otimes C)\bar{e}$. The other subsystem is given by

$$\begin{aligned} \epsilon\dot{e}^* &= (A + BF + \epsilon R_e)e^* - BF_1^* z_1^* - BF_2^* z_2^* + \epsilon\,\mathbb{G}_e^* w \\ \epsilon\dot{z}_1^* &= \epsilon C_1\bar{K}z_1^* + C_1 z_2^* + \epsilon\,\mathbb{G}_{z1}^* w \\ \dot{z}_2^* &= \epsilon\widetilde{E}_z z_1^* + \widetilde{A}_z z_2^* + \epsilon\,\mathbb{G}_{z2}^* w \end{aligned}$$

When all agents have reached an agreement, $\boldsymbol{\zeta}$ is zero, which imposes no constraints on (e^*, z_1^*, z_2^*). In fact, (e^*, z_1^*, z_2^*)-subsystem determines the consensus trajectories when $\boldsymbol{\zeta} = 0$. It enunciates the fact that the consensus trajectories may be unbounded. It is inferred that the objective in synchronization of networked systems is to design a protocol such that the reduced-order dynamics (15) vanishes in time.

Disturbance Rejection Properties: We demonstrate that $\|T_{w\zeta}\|_\infty$ shrinks as ϵ reduces. Let $\|\boldsymbol{\zeta}\| \leq \rho_\zeta\|\bar{e}\|$ for some ρ_ζ independent of ϵ. As $A + BF$ is Hurwitz stable, there exists $P_c = P_c^{\mathrm{T}} > 0$ which solves

$$(\mathrm{I}_{\bar{N}} \otimes (A + BF))^{\mathrm{T}} P_c + P_c\,(\mathrm{I}_{\bar{N}} \otimes (A + BF)) = -2\mathrm{I}_{\bar{N}pn_q}$$

Denote $s_0 \geq \|P_c(\mathrm{I}_{\bar{N}} \otimes R_e)\|$, $s_1 \geq \|P_c(\mathrm{I}_{\bar{N}} \otimes BF_1^*)\|$, $s_2 \geq \|P_c(\mathrm{I}_{\bar{N}} \otimes BF_2^*)\|$ and $\rho_1 \geq \|P_c\widehat{\mathbb{G}}_e\|$, all bounded. Therefore, there exists an $\epsilon_{11} \in (0,1]$ such that, for every $\epsilon \in (0, \epsilon_{11}]$, the inequality $(1 - 2\epsilon s_0) > 0$ holds. Let $W_c = \epsilon\bar{e}^{\mathrm{T}} P_c \bar{e}$. Differentiation yields

$$\dot{W}_c \leq -\|\bar{e}\|^2 + 2\rho_2\sqrt{\|\bar{z}_1\|^2 + \|\bar{z}_2\|^2}\|\bar{e}\| + 2\epsilon\rho_1\|\bar{e}\|\|\boldsymbol{w}\|$$

where $\rho_2 \geq \sqrt{2}\max\{s_1, s_2\}$. As $\mathcal{L} \in \mathcal{G}_\beta$, $\lambda(\Delta) \in \mathbb{C}^+$. Therefore, since $K_1 < 0$, $\Delta \otimes K_1$ is Hurwitz stable. Thus, for any $\mathcal{G}_\beta$, there exists a $P_1 > 0$ such that the following holds

$$(\Delta \otimes K_1)^{\mathrm{H}} P_1 + P_1(\Delta \otimes K_1) = -Q_1 < 0 \tag{16}$$

Proposition 1: Given $\mathcal{G}_\beta$, replacing

$$P_1 = -\operatorname{diag}\{p_1, \cdots, p_{\bar{N}}\} \otimes K_1^{-1}$$

where $p_{\bar{N}} = 1$ and $p_i = \frac{\beta^2}{9} p_{i+1}$, in (16) gives rise to $Q_1 > 4\mathsf{q}\mathrm{I}$ where $\mathsf{q} > 0$ depends on β. It implies $\|P_1\|$ is bounded.

Proof: See [20]. ∎

It is worth noting that P_1 is found for a set of networks, say $\mathcal{G}_\beta$, not for a given network $\mathcal{L}$. Choose $W_1 = \mathsf{q}\epsilon\bar{z}_1^{\mathrm{T}} P_1 \bar{z}_1$. An upper bound for $\dot{W}_1$ is then given by

$$\begin{aligned}\dot{W}_1 \leq\ & -2\mathsf{q}^2\|\bar{z}_1\|^2 - 2\mathsf{q}\|\bar{z}_1\|^2(\mathsf{q} - \epsilon s_3) \\ & + 2\mathsf{q}s_4\|\bar{z}_1\|\|\bar{z}_2\| + 2\mathsf{q}\epsilon\rho_3\|\bar{z}_1\|\|\boldsymbol{w}\|\end{aligned}$$

where $s_3 \geq \|P_1(\mathrm{I}_{\bar{N}} \otimes C_1\bar{K})\|$, $s_4 \geq \|P_1(\mathrm{I}_{\bar{N}} \otimes C_1)\|$, and $\rho_3 \geq \|P_1\widehat{\mathbb{G}}_{z1}\|$. Since $\widetilde{A}_z$ is Hurwitz stable, the equation

$$(\mathrm{I}_{\bar{N}} \otimes \widetilde{A}_z)^{\mathrm{T}} P_2 + P_2(\mathrm{I}_{\bar{N}} \otimes \widetilde{A}_z) = -(2\mathsf{q} + \mathsf{q}^{-1}s_4^2)\mathrm{I}_{\bar{N}\bar{p}}$$

has the solution $P_2 = P_2^{\mathrm{T}} > 0$. The derivative of $W_2 = \mathsf{q}\,\bar{z}_2^{\mathrm{T}} P_2 \bar{z}_2$ along the trajectories of (15c) is bounded by

$$\dot{W}_2 \leq -(2\mathsf{q}^2 + s_4^2)\|\bar{z}_2\|^2 + 2\mathsf{q}\epsilon s_5\|\bar{z}_2\|\|\bar{z}_1\| + 2\mathsf{q}\epsilon\rho_4\|\bar{z}_2\|\|\boldsymbol{w}\|$$

in which $s_5 \geq \|P_2(\mathrm{I}_{\bar{N}} \otimes \widetilde{E}_z)\|$ and $\rho_4 \geq \|P_2\widehat{\mathbb{G}}_{z2}\|$. Consider $W_o = W_1 + W_2$ and differentiate it in time. One may find an upper bound for $\dot{W}_o$ as

$$\begin{aligned}\dot{W}_o \leq\ & -\mathsf{q}^2\|\bar{z}_1\|^2 + 2\mathsf{q}s_4\|\bar{z}_1\|\|\bar{z}_2\| - s_4^2\|\bar{z}_2\|^2 \\ & - 2\mathsf{q}\|\bar{z}_1\|^2(\mathsf{q} - \epsilon s_3) + 2\mathsf{q}\epsilon s_5\|\bar{z}_2\|\|\bar{z}_1\| - \mathsf{q}^2\|\bar{z}_2\|^2 \\ & - \mathsf{q}^2\|\bar{z}_1\|^2 - \mathsf{q}^2\|\bar{z}_2\|^2 + 2\mathsf{q}\epsilon\rho_3\|\bar{z}_1\|\|\boldsymbol{w}\| + 2\mathsf{q}\epsilon\rho_4\|\bar{z}_2\|\|\boldsymbol{w}\|\end{aligned}$$

The first line is non-positive. There exists an $\epsilon_{22} \in (0,1]$ such that for every, $0 < \epsilon \leq \epsilon_{22}$, $\begin{bmatrix} 2(\mathsf{q} - \epsilon s_3) & -\epsilon s_5 \\ -\epsilon s_5 & \mathsf{q} \end{bmatrix} > 0$. Then, for every $\epsilon \in (0, \epsilon_{22}]$, it turns out that

$$\dot{W}_o \leq -\mathsf{q}^2\|\bar{z}_1\|^2 - \mathsf{q}^2\|\bar{z}_2\|^2 + 2\mathsf{q}\epsilon\rho_5\sqrt{\|\bar{z}_1\|^2 + \|\bar{z}_2\|^2}\|\boldsymbol{w}\|$$

where $\rho_5 \geq \sqrt{2}\max\{\rho_3, \rho_4\}$. Choose $V = (2 + \rho_\zeta^2)W_c + (1 + (2+\rho_\zeta^2)^2\rho_2^2\,\mathsf{q}^{-2})W_o$ as the Lyapunov function candidate for the system (15). Let $\epsilon_1^* = \min\{\epsilon_{11}, \epsilon_{22}\}$. For every $\epsilon \in (0, \epsilon_1^*]$, the time derivative of V satisfies

$$\begin{aligned}\dot{V} \leq & -\|\bar{e}\|^2 + 2(2 + \rho_\zeta^2)\rho_2\sqrt{\|\bar{z}_1\|^2 + \|\bar{z}_2\|^2}\|\bar{e}\| \\ & - (2 + \rho_\zeta^2)^2\rho_2^2(\|\bar{z}_1\|^2 + \|\bar{z}_2\|^2) \\ & - \rho_\zeta^2\|\bar{e}\|^2 - (\mathsf{q}^2\|\bar{z}_1\|^2 + \mathsf{q}^2\|\bar{z}_2\|^2 + \|\bar{e}\|^2) \\ & + 2\epsilon\rho_6\sqrt{\mathsf{q}^2\|\bar{z}_1\|^2 + \mathsf{q}^2\|\bar{z}_2\|^2 + \|\bar{e}\|^2}\|\boldsymbol{w}\|\end{aligned} \tag{17}$$

where $\rho_6 \geq \sqrt{2}\max\{\rho_1(2 + \rho_\zeta^2), \rho_5(1 + (2 + \rho_\zeta^2)^2\rho_2^2\,\mathsf{q}^{-2})\}$. The first two lines form a non-positive term. Completing the square using the last two lines, we arrive at

$$\dot{V} + \|\boldsymbol{\zeta}\|^2 - (\epsilon\rho_6)^2\|\boldsymbol{w}\|^2 \leq 0 \tag{18}$$

From Kalman-Yakubovich-Popov Lemma (see e.g. [21]), it follows that $\|T_{w\zeta}\|_\infty \leq \epsilon\rho_6$. Obviously, from (17), it is observed that $(\bar{e}, \bar{z}_1, \bar{z}_2)$ is globally exponentially stable at the origin when $\boldsymbol{w} = 0$.

So far, we have shown that the proposed protocols reject $\boldsymbol{w}$ from $\boldsymbol{\zeta}$ to any arbitrary level. That is, the $\mathcal{H}_\infty$ norm of the corresponding transfer function can be made arbitrarily small by squeezing ϵ. Lemma 3 shows that the proposed protocols (7) has a similar decoupling effect on $\mathfrak{e}$, and every individual mutual disagreement $\mathfrak{e}_{i,j}$ can be made arbitrarily small. We define $\mathfrak{e}_{i,j} = T_{w\mathfrak{e}}^{i,j}(s)\boldsymbol{w}$.

Lemma 3: Let $\|T_{w\zeta}\|_\infty \leq \epsilon\sigma'$ for some $\sigma' > 0$ and sufficiently small $\epsilon \in (0,1]$. There exists a positive constant σ such that $\|T_{w\mathfrak{e}}^{i,j}\|_\infty \leq \epsilon\sigma$.

Proof: By an appropriate choice of $\mathbb{A}$ and $\mathbb{B}$, the relation between the output $\boldsymbol{\zeta}$ and the input $\boldsymbol{w}$ can be described as

$$\boldsymbol{\zeta} = \epsilon(L \otimes C)(s\mathrm{I} - \mathbb{A})^{-1}\mathbb{B}\boldsymbol{w}$$

We pick one agent arbitrarily. Let it be agent N. As $L\mathbf{1} = 0$, it holds that $\sum_{j=1}^N l_{ij} y_N = 0$.

Thus, we have

$$\zeta_i = \sum_{j=1}^N l_{ij}y_j - \sum_{j=1}^N l_{ij}y_N = \sum_{j=1}^N l_{ij}\mathfrak{e}_{j,N}$$

Define $\sigma_i \triangleq \zeta_i - \zeta_N$, $\boldsymbol{\sigma} \triangleq \operatorname{col}\{\sigma_i\}$, and $\mathfrak{e}_{\boldsymbol{N}} \triangleq \operatorname{col}\{\mathfrak{e}_{i,N}\}$ for $i \in \mathbb{S}_1$ where $\mathbb{S}_1 = \{1, 2, \cdots, N-1\}$. Then, one may find $\sigma_i = \sum_{j=1}^{N-1} l_{ij}^* \mathfrak{e}_{j,N}$ where $l_{ij}^* = l_{ij} - l_{Nj}$ for $j \in \mathbb{S}$. Let $\bar{L} \triangleq [l_{ij}^*]$ for $i \in \mathbb{S}_1$, $j \in \mathbb{S}$ be obtained by removing the last row of $L - \mathbf{1}\,l_N^{\mathrm{T}}$ where l_k^{T} denotes the k-th row of L.

Let $L^* \triangleq [l_{ij}^*]$ for $i \in \mathbb{S}_1$, $j \in \mathbb{S}_1$ be the reduced Laplacian which is found by discarding the last column of $\bar{L}$. According to [22], L^* is non-singular. From the definition of L^*, it is obtained that $\boldsymbol{\sigma} = (L^* \otimes \mathrm{I}_p)\mathfrak{e}_{\boldsymbol{N}}$. On the other hand, from the definition of σ_i, it follows that $\boldsymbol{\sigma} = \epsilon\,(\bar{L} \otimes C)(s\mathrm{I} - \mathbb{A})^{-1}\mathbb{B}\boldsymbol{w}$. Hence,

$$\mathfrak{e}_{\boldsymbol{N}} = \epsilon\,(L^* \otimes \mathrm{I}_p)^{-1}(\bar{L} \otimes C)(s\mathrm{I} - \mathbb{A})^{-1}\mathbb{B}\boldsymbol{w}$$

It shows that $\|T_{w\mathfrak{e}}^{i,j}\|_\infty$ is of order ϵ, and there exists a constant $\sigma > 0$ such that $\|T_{w\mathfrak{e}}^{i,j}(s)\|_\infty < \epsilon\sigma$. ∎

Therefore, every mutual disagreement can be made arbitrarily small. In other words, given $\gamma > 0$, there exists an $\epsilon_2^* \in (0, \epsilon_1^*]$ such that, for every $\epsilon \in (0, \epsilon_2^*]$, the closed-loop transfer function from $\boldsymbol{w}$ to $\mathfrak{e}$ satisfies $\|T_{w\mathfrak{e}}(s)\|_\infty < \gamma$. Hence, any desired accuracy of synchronization can be obtained.

We point out that the problem is solved for a given set of networks $\mathcal{G}_\beta$, not for a specific network $\mathcal{L}$.