Almost Synchronization for Non-Identical Introspective Multi-Agent Systems Under External Disturbances

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Abstract

This paper brings up the notion of “$H_\infty$ almost synchronization” for multi-agent systems under directed interconnection structures where agents are linear, right-invertible and introspective with non-identical dynamics and subject to external disturbances. The objective is to suppress the impact of disturbances on the synchronization error dynamics in terms of the $H_\infty$ norm of the corresponding closed-loop transfer function. To reach the goal, designing local dynamic compensators using the agents’ self measurements, the network of non-identical agents is converted to a network of almost identical agents which mimics the dynamic behavior of a desired system; then, inspired by the concept of almost disturbance decoupling for a single agent using the singular perturbation method, an observer-based parameterized protocol is introduced to achieve synchronization with the prescribed $H_\infty$ performance. In addition, the problem of regulating the consensus trajectories to a reference signal is addressed. The application of the proposed method to formation control is furthermore elucidated. Simulation results are provided to illustrate the method.

Key words: Synchronization; Disturbance rejection; Interconnection networks; Multi-agent systems.

1 INTRODUCTION

Dynamical networked systems have received a great deal of attention in the past decade. The application is widespread and span formations of unmanned space, land, and marine vehicles, sensor networks, coordinated decision making, etc. (see Ren et al., 2005; Olfati-Saber et al., 2007). The objective is state (output) synchronization; i.e. the states (outputs) of all the agents have to converge to a common trajectory.

The seminal works of (Wu and Chua, 1995a) and (Wu and Chua, 1995b) have substantially contributed in analysis and design of multi-agent systems; in particular, Wu and Chua (1995a) introduce the application of the graph theory to represent the communication topology; Wu and Chua (1995b) provide a systematic approach for stability analysis of multi-agent systems by virtue of the Kronecker product. Many papers consider consensus of first order or second order integrator dynamics; (e.g., Olfati-Saber and Murray, 2004; Ren and Beard, 2005). They basically rely on full state information from the network and design static decentralized protocols.

State consensus of general linear agents is addressed in (Fax and Murray, 2004; Seo et al., 2009; Li et al., 2010; Zhang et al., 2011; Yang et al., 2011c) where partial-state information is given to each agent via the network and dynamic protocols are introduced. Seo et al. (2009) propose a low-gain approach by filtering the information that each agent receives whereas Fax and Murray (2004) consider self-feedback for all agents. A significant breakthrough in the design of dynamic protocols is presented by Li et al. (2010) where conventional observers are expanded to distributed observers while agents are capable of exchanging information about their own estimates over the network. The result is extended to LQR-based...
optimal design by Zhang et al. (2011). More general networks and regulation of output consensus are studied by Yang et al. (2011c).

Networks of non-identical agents have not yet been investigated thoroughly. The common assumption is that agents are introspective; that is, agents possess some knowledge about their own states. For networks of nonlinear agents, Zhao et al. (2011) present criteria for state consensus. Output consensus for weakly minimum-phase systems of relative degree one is studied in (Chopra and Spong, 2008) where local feedbacks are utilized to decouple zero dynamics and create a single integrator system. Embedding additional models within agents, Kim et al. (2011) propose a controller for SISO minimum-phase systems. Yang et al. (2011b) develop a method to represent a non-identical multi-agent system as a network of asymptotically identical agents, and design a decentralized protocol to reach output consensus. Relaxing the self-knowledge (introspective) assumption, Zhao et al. (2010) propose a protocol for networks of non-identical and passive nonlinear systems. Also, (Grip et al., 2012) puts forward a distributed dynamic protocol for networks of linear, non-identical and non-introspective agents.

1.1 Contribution of The Paper

This paper studies the $\mathcal{H}_\infty$ disturbance rejection problem for a group of linear, right invertible agents with non-identical dynamics of any order subject to external disturbances and under directed interconnection topologies. To be more explicit, we aim to construct a family of parameterized linear time-invariant protocols based on a distributed observer such that i) synchronization is accomplished in the absence of disturbance, and ii) disturbance is attenuated in the controlled output (which is selected as a function of the disagreement between any pair of agents) to any arbitrarily small value in the sense of the $\mathcal{H}_\infty$ norm of the transfer function.

The paper brings forth the notion of “$\mathcal{H}_\infty$ almost synchronization”. The proposed method facilitates regulation of consensus trajectories to a given reference and allows for bringing in the notion of “$\mathcal{H}_\infty$ almost regulation of output consensus”. To complete the study, the concept of “$\mathcal{H}_\infty$ almost formation” is also introduced.

1.1.1 Previous Work

Synchronization in the presence of external disturbances and uncertainty has been the topic of fairly few papers which usually focus on networks of identical agents. Robust synchronization of a network is analyzed in (Tanner et al., 2004) where leader-to-formation stability builds on the notion of input-to-state stability for single agent and is used to assess robustness of followers with respect to the leader’s input. Consensus of a network of $N$ linear agents with identical dynamics under an undirected communication topology is explored in (Li et al., 2010) where it is demonstrated that the problem of model-reference output consensus with the prescribed $\mathcal{H}_\infty$ performance is solvable if $\mathcal{H}_\infty$ control of $N$ independent systems with the same dimension as the agents is solvable using a decentralized protocol. In addition, Li et al. (2009) show that considering the outputs of agents as the controlled output, for leaderless consensus, the $\mathcal{H}_\infty$ performance of the network cannot be better than the $\mathcal{H}_\infty$ performance of one isolated agent. An LMI-based protocol together with the concept of pinning is proposed by Li et al. (2009) for undirected networks of identical agents.

For a directed network of scalar first-order and second-order integrator systems under external disturbances and graph Laplacian uncertainty, Lin et al. (2008) and Lin and Jia (2010) propose $\mathcal{H}_\infty$ controllers based on LMI. For identical agents, Ugrinovskii (2011) seeks a distributed consensus observer with $\mathcal{H}_\infty$ robust performance using the concept of vector dissipativity.

As for multi-agent systems with non-identical dynamics, contrary to (Li et al., 2009, 2010), the control problem cannot be broken down into $\mathcal{H}_\infty$ control of $N$ independent systems. We intend to deal with this situation and propose a method to reach any arbitrary $\mathcal{H}_\infty$ performance.

1.1.2 Methodology

We utilize the time-scale structure assignment technique (Ozcan et al., 1992) rooted in the methodology of singular perturbation (Kokotovic et al., 1986) in order to propose a method which provides a family of linear time-invariant protocols parameterized in terms of a tuning parameter $\epsilon$. However, the structure of the protocols is independent from the parameter $\epsilon$, and one can develop the protocol structure at one stage and tune the parameter $\epsilon$, in a certain range, in order to reach the desired level of disturbance rejection. Since the structure is continuous in the parameter $\epsilon$, tuning may be even carried out online. Hence, the proposed method to solve $\mathcal{H}_\infty$ almost synchronization for multi-agent systems is not iterative, but it provides a one-shot design. In contrast to (Ozcetin et al., 1992), the analysis presented in this paper consists in the Lyapunov stability theory.

To pursue the objective, first, a local output feedback along with pre-compensators is designed for each agent so that all agents are shaped into a particular form and a network of almost identical agents with various disturbance matrices emerges. Shaping, which plays a pivotal role in solvability of the problem, is viable because agents are introspective that means they have partial knowledge about their own states. Then, assuming that
agents can transmit information using the same network topology, the dynamic protocol is derived to attain the desired \( H_\infty \) performance.

The main focus of this note is on the leaderless syn-
chronization in which the aim is to make the outputs of agents synchronous although the consensus trajectory is not controlled. Afterwards, the approach is expanded to solve “\( H_\infty \) almost regulation of output consensus” where the objective is to find a family of parameterized dynamic protocols such that i) output synchronization with respect to a desired reference is accomplished in the absence of disturbance, and ii) disturbance is attenuated in the controlled output (which is selected as the disagreement between the output of each agent and the reference) to any arbitrarily small value in the sense of the \( H_\infty \) norm of the transfer function. The method is, moreover, broadened to encompass “\( H_\infty \) almost formation”, where the agents are asked to maintain their relative outputs as desired while they may be asked simultaneously to regulate the outputs with a reference.

### 2 Notations and Preliminaries

Throughout the paper, \( I_n \) denotes the identity matrix of dimension \( n \times n \) and \( 1_n \in \mathbb{R}^n \) means a vector whose entries are all one; when clear from the context, the subscript is dropped. 0 may show scalar, vector or matrix of appropriate dimension. \( A^T \), \( AH \) and \( \|A\| \) indicate transpose, conjugate transpose, and induced 2-norm of the given matrix \( A \). The element \((i,j)\) of \( A \) is shown by \( a_{ij} \); thus, \( A = [a_{ij}] \). For \( A \in \mathbb{R}^{n \times m} \) and \( B \in \mathbb{R}^{p \times q} \), \( A \otimes B \) symbolizes the Kronecker product which is a \( np \times mq \) matrix:

\[
A \otimes B = \begin{bmatrix}
a_{11}B & \cdots & a_{1m}B \\
\vdots & \ddots & \vdots \\
a_{n1}B & \cdots & a_{nm}B
\end{bmatrix}
\]

The open left-half complex plane is represented by \( \mathbb{C}^- \) whereas \( \mathbb{C}^+ \) indicates the open right-half complex plane. \( \text{diag}\{A, B\} \) means a block diagonal matrix constructed by \( A \) and \( B \). Likewise, \( \text{diag}\{A_i\} \) for \( i = 1, \cdots, n \) means \( \text{diag}\{A_1, \cdots, A_n\} \). Also, \( x = \text{col}\{x_i\} \) for \( i = 1, \cdots, n \) is adopted to denote \( x = [x_1^T, \cdots, x_n^T]^T \). The real part of a complex number \( \lambda \) is represented by \( \text{Re}\{\lambda\} \). For a transfer function \( T(s) \), the \( H_\infty \) norm is denoted \( \|T(s)\|_\infty \).

Let \( \mathcal{L} \) be a weighted directed graph with \( n \) nodes. If there is an edge from node \( j \) to node \( i, a_{ij} > 0, a_{ij} \in \mathbb{R} \) is assigned to the edge; otherwise, \( a_{ij} = 0 \). If the graph is not allowed to have self-loops, \( a_{ii} = 0 \). \( \mathcal{A} := [a_{ij}] \) is the weighted adjacency matrix of \( \mathcal{L} \). We say that there is a directed path from node \( i_k \) to node \( i_m \) if there is a sequence of distinct nodes such that \( a_{i_{k+1}i_k}, a_{i_ki_m} > 0 \) for \( k = 1, \cdots, m - 1 \). A subgraph \( \mathcal{L}_s \) of \( \mathcal{L} \) is a directed tree if every node of \( \mathcal{L}_s \) has exactly one incoming edge, except one distinguished node called the root node with no incoming edge. If the directed tree contains all the nodes of \( \mathcal{L} \), it is said to be a directed spanning tree, and there is a directed path from the root to every other node of \( \mathcal{L} \).

The Laplacian of \( \mathcal{L} \) is denoted by \( L = [l_{ij}] \) where \( l_{ii} = \sum_{j=1}^n a_{ij} \) and \( l_{ij} = -a_{ij} \) for \( i \neq j \). It implies that the sum of the elements on each row of \( L \) is zero; i.e. \( \sum_{j=1}^n l_{ij} = 0 \). Thus, \( 1_n \) is a right eigenvector of \( L \) associated with the eigenvalue at zero. In accordance with (Ren and Beard, 2005), if \( \mathcal{L} \) contains a directed spanning tree, \( L \) has a simple eigenvalue at zero and all the other eigenvalues are in \( \mathbb{C}^+ \).

### 3 Multi-Agent Systems

A multi-agent system is referred to a network of multiple-input multiple-output agents described by linear time-invariant models as

\[
\text{Agent } i : \begin{cases}
\dot{x}_i = A_i \bar{x}_i + B_i \bar{u}_i + G_i \bar{w}_i \\
y_i = C_i \bar{x}_i
\end{cases}
\]

in which \( i \in \mathcal{S} := \{1, \cdots, N\} \), \( \bar{x}_i \in \mathbb{R}^{n_i}, \bar{u}_i \in \mathbb{R}^{m_i}, y_i \in \mathbb{R}^p, \bar{w}_i \in \mathcal{L}_\infty \) are the agent’s state, control input, output, and external disturbance, respectively. Agents are introspective; thus, each agent possesses partial knowledge of its own states through the local measurement:

\[
y_{m,i} = C_{m,i} \bar{x}_i
\]

Furthermore, agents are allowed to exchange information using the network’s communication topology which is described by a directed graph \( \mathcal{L} \), with no self loops, associated with the adjacency matrix \( A_\mathcal{L} = [a_{ij}] \) and the Laplacian matrix \( L = [l_{ij}] \).

Each agent has access to a weighted linear combination of outputs. In particular, the network measurement given to agent \( i \in \mathcal{S} \) is:

\[
\zeta_i = \sum_{j=1}^N a_{ij} (y_i - y_j) = \sum_{j=1}^N l_{ij} y_j
\]

In addition, it is assumed that agents can exchange additional information over the network. The transmission of additional information conforms with the network topology and facilitates the design of a distributed observer for the network. Thus, agent \( i \) has access to the following
quantity:
\[ \hat{\zeta}_i = \sum_{j=1}^{N} l_{ij} \eta_j \]  
(1e)
where \( \eta_j \in \mathbb{R}^p \) will be specified later as a dynamic protocol is introduced.

**Definition 1** For given \( \beta > 0 \) and integer \( N_0 \geq 1 \), \( G_\beta \) is the set of graphs composed of \( N \) nodes where \( N \leq N_0 \) such that every \( L \in G_\beta \) has a directed spanning tree and the eigenvalues of its Laplacian, denoted \( \lambda_1, \cdots, \lambda_N \), satisfy
\[ \text{Re}\{\lambda_i\} > \beta, \quad \lambda_i \neq 0 \]

According to (Ren and Beard, 2005), the Laplacian \( L \) associated with \( L \in G_\beta \) has a simple eigenvalue at zero and the rest are located in \( \mathbb{C}^+ \).

**Assumption 1** We make the following assumptions for agent \( i \in S \).

1. \((A_i, B_i, C_i)\) is right-invertible;
2. \((A_i, B_i)\) is stabilizable and \((A_i, C_i)\) is detectable;
3. \((A_i, C_{m,i})\) is detectable;

\[ \beta > 0 \] is taken into consideration, and it is demonstrated how to find, if possible, a family of time-invariant dynamic protocols which satisfy the above. The problem of the \( \mathcal{H}_\infty \) almost synchronization and almost synchronization is solved; specifically, there exists a family of linear time-invariant dynamic protocols, parameterized in terms of a tuning parameter \( \epsilon \in (0, 1] \), of the form
\[ \begin{aligned}
\dot{x}_i &= A_i(\epsilon) x_i + B_i(\epsilon) \col \{ \zeta_i, \zeta_m, y_m \} \\
\dot{u}_i &= C_i(\epsilon) x_i + D_i(\epsilon) \col \{ \zeta_i, \zeta_m, y_m \}
\end{aligned} \]
(4)
where \( x_i \in \mathbb{R}^n \) and \( i \in S \) such that

\( (i) \) for any \( \beta > 0 \), there exists \( \epsilon_1 \in (0, 1] \) such that if \( \epsilon \in (0, \epsilon_1] \), synchronization is accomplished in the absence of disturbance; i.e. \( \forall \epsilon \in (0, \epsilon_1] \) when \( w = 0 \)
\[ e_{i,j} = y_i - y_j \rightarrow 0, \quad \forall i, j \in S \quad \text{as} \quad t \rightarrow \infty \]

\( (ii) \) for any given \( \gamma > 0 \), there exists an \( \epsilon_2 \in (0, \epsilon_1] \) such that if \( \epsilon \in (0, \epsilon_2] \), the closed-loop transfer function from \( w \) to \( e \) satisfies
\[ \| T_{we}(s) \|_\infty < \gamma \]

4.2 Result 1: \( \mathcal{H}_\infty \) Almost Synchronization

**Theorem 1** Under Assumption 1 and for the set \( G_\beta \), the problem of the \( \mathcal{H}_\infty \) almost synchronization is solvable; specifically, there exists a family of linear time-invariant dynamic protocols, parameterized in terms of a tuning parameter \( \epsilon \in (0, 1] \), of the form
\[ \begin{aligned}
\dot{x}_i &= A_i(\epsilon) x_i + B_i(\epsilon) \col \{ \zeta_i, \zeta_m, y_m \} \\
\dot{u}_i &= C_i(\epsilon) x_i + D_i(\epsilon) \col \{ \zeta_i, \zeta_m, y_m \}
\end{aligned} \]
(4)
where \( x_i \in \mathbb{R}^n \) and \( i \in S \) such that

\( (i) \) for any \( \beta > 0 \), there exists an \( \epsilon_1 \in (0, 1] \) such that if \( \epsilon \in (0, \epsilon_1] \), synchronization is accomplished in the absence of disturbance; i.e. \( \forall \epsilon \in (0, \epsilon_1] \) when \( w = 0 \)
\[ e_{i,j} = y_i - y_j \rightarrow 0, \quad \forall i, j \in S \quad \text{as} \quad t \rightarrow \infty \]

\( (ii) \) for any given \( \gamma > 0 \), there exists an \( \epsilon_2 \in (0, \epsilon_1] \) such that if \( \epsilon \in (0, \epsilon_2] \), the closed-loop transfer function from \( w \) to \( e \) satisfies
\[ \| T_{we}(s) \|_\infty < \gamma \]

The proof of Theorem 1 is presented in the subsequent sections in a constructive way. It is shown that Theorem 1 follows directly from Lemmas 1 and 2.

In the following, first, we present the solution for a subgroup of multi-agent systems of the form (1), and then we explain how to generalize the method.

4.3 Special Case

In this section, a special case of multi-agent system (1) is taken into consideration, and it is demonstrated how a family of dynamic protocols given by (4) achieves \( \mathcal{H}_\infty \) almost synchronization.
Consider a network of agents of the form
\[
\begin{align*}
\dot{x}_i &= Ax_i + B(u_i + Rx_i) + E_i w_i \\
y_i &= Cx_i
\end{align*}
\] (5a)
\begin{align*}
0 &\quad \text{where } x_i \in \mathbb{R}^{p_{n_i}}, u_i, y_i \in \mathbb{R}^p, w_i \in \mathbb{R}^{n_i}, R \in \mathbb{R}^{p \times p_{n_i}}, E_i \in \mathbb{R}^{p_{n_i} \times n_i} \text{ and }
\end{align*}
where
\[
A = \begin{bmatrix} 0 & I_{p(n_n-1)} \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ E_1 \end{bmatrix}, C = \begin{bmatrix} I_p \\ 0 \end{bmatrix}
\] (6)
in which \(n_n \geq 1\) is an integer. Each agent collects the network measurements (1d) and (1e). Let us partition the system matrices such that
\[
A = \begin{bmatrix} 0_{p \times p} & C_1 \\ 0_{p(n_n-1) \times p} & A_1 \end{bmatrix}, B = \begin{bmatrix} 0_{p \times p} \\ B_1 \end{bmatrix}, E_i = \begin{bmatrix} E_{1,i} \\ E_{2,i} \end{bmatrix}
\]
where \(C_1 = [I_p, 0, \cdots, 0] \in \mathbb{R}^{p \times (n_n-1)}\) and \(E_{1,i} \in \mathbb{R}^{p \times n_i}\). Also consider \(R = [R_1, R] \in \mathbb{R}^{p \times p}\).

### 4.3.1 Design Procedure

For each agent \(i \in S\), we construct the observer-based protocol of the form
\[
\begin{align*}
\dot{\hat{x}}_i &= A\hat{x}_i + B(u_i + R\hat{x}_i) - \epsilon^{-1}K(\zeta_i - \hat{\zeta}_i) \\
\epsilon \quad &\text{where } \hat{\zeta}_i \in \mathbb{R}^{p_{n_i}} \text{ and } \epsilon \in (0, 1] \text{ is the tuning parameter. } \\
0 &\text{\hat{\zeta}_i \text{ is the additional information transmitted over the network and given by (1e) where } } n_i = C\hat{x}_i. \text{ We define the matrix } S \text{ as }
\end{align*}
\] (7a)
\begin{align*}
\epsilon^{-1}K(\zeta_i - \hat{\zeta}_i)
\end{align*}
(7b)
where \(\hat{x}_i \in \mathbb{R}^{p_{n_i}}\) and \(\epsilon \in (0, 1]\) is the tuning parameter, \(\hat{\zeta}_i\) is the additional information transmitted over the network and given by (1e) where \(n_i = C\hat{x}_i\). We define the matrix \(S\) as
\[
S \triangleq \text{diag}(I_p, \epsilon I_p, \cdots, \epsilon^{n_n-1}I_p) \in \mathbb{R}^{p_{n_n} \times p_{n_n}}
\] (8)
and consider the following structure for the the observer gain:
\[
K = \begin{bmatrix} K_1 \\ \bar{K}K_1 \end{bmatrix}
\] (9)
in which \(K_1 \in \mathbb{R}^{p \times p}\), \(\bar{K} \in \mathbb{R}^{p \times (n_n-1) \times p}\). The protocol gains are chosen as follows:

- considering the controllable pair \((A, B)\), choose \(F\) such that \(A + BF\) is Hurwitz stable;
- considering the observable pair \((A_1 + B_1\bar{R}, C_1)\), choose \(K\) such that \(A_1 \triangleq A_1 + B_1\bar{R} - \bar{K}C_1\) is Hurwitz stable;
- choose \(K < 0\).

**Lemma 1** For any given \(\gamma, \beta > 0\), the dynamic protocol (7) solves the \(\mathcal{H}_\infty\) almost synchronization problem for multi-agent systems of the form (5) with the communication topology \(\mathcal{G} \in \mathcal{G}_\beta\).

**PROOF.** See Appendix A.

The protocol (7) is inspired by the concept of \(\mathcal{H}_\infty\) almost disturbance decoupling for a single agent proposed by Ozbay et al. (1992). Controller (7) is, in fact, developed from the special coordinated basis (s.c.b) representation of the triple \((A, B, C)\) and the triples \((A, E_i, C)\).

**Remark 1** The network of agents (5) resembles a network of identical agents with different disturbance matrices. Having non-identical disturbance matrices significantly complicates \(\mathcal{H}_\infty\) analysis for the network. Also, it is not possible to convert the problem into a decentralized \(\mathcal{H}_\infty\) control problem as presented by Li et al. (2009, 2010) where undirected communication topologies are considered. However, the proposed design method solves the problem using a parameterized protocol which achieves synchronization with any arbitrarily small \(\mathcal{H}_\infty\) gain while the order of the controller is fixed and the communication topology is directed.

#### 4.4 Almost Identical Representation for Networks of Non-Identical Agents

In this section, we present a method to shape a multi-agent system of the form (1) into the form (5). The requisite for shaping is the local measurement (1c). Therefore, the result given in the preceding section is expanded to the general form of non-identical agents.

To achieve the aim, we propose a three-step procedure as Yang et al. (2011b). Fig. 2 shows a graphical interpretation of the procedure and is illustrative. The first step is to square down each agent. Specifically, an asymptotically stable, minimal order compensator is designed according to (Saberi and Sannuti, 1988) and cascaded with the agent. The resulting system is invertible and square. The pre-compensator preserves stabilizability, detectability, and stability of the zero dynamics as well as the infinite-zero structure of the original system. It introduces additional invariant zeros which are placed in \(\mathbb{C}^-\).

**Definition 3** \(n_{q0} \geq 1\) is the maximum order of the infinite zeros of all triples \((A_i, B_i, C_i)\) in a multi-agent system.

In the second step, a rank-equalizing pre-compensator is designed according to Saberi et al. (1990) such that the resulting system has a uniform rank \(n_q \geq n_{q0}\).

Finally, an output feedback is fed to the system so as to decouple the zero dynamics from the infinite-zero structure such that the infinite-zero structure adopts the behavior dynamics of a desired invertible system of the uniform rank \(n_q\) with no invariant zeros. Accordingly, each
agent can be described by
\[
\dot{x}_i = Ax_i + B(u_i + Rx_i) + E_{d,i}\tilde{w}_i + B\rho_i \\
y_i = Cx_i
\] (10a)
(10b)
where \(u_i = Mu'_i\), \(u_i, y_i \in \mathbb{R}^p\); \(A, B\) and \(C\) are given by (6); \(R \in \mathbb{R}^{p \times p(n_u - 1)}\) and non-singular \(M \in \mathbb{R}^{p \times p}\) are chosen arbitrarily. Also, \(\rho_i \in \mathbb{R}^p\) is evolved from the exponentially stable system described by
\[
\dot{\tilde{x}}_i = H_i\tilde{x}_i + E_{o,i}\tilde{w}_i \\
\rho_i = W_i\tilde{x}_i
\] (11a)
(11b)

\(E_{d,i}, E_{o,i}, H_i,\) and \(W_i\) are given in the proof of Lemma 2 which formally states the result.

**Lemma 2** Consider the multi-agent system (1) satisfying Assumption 1. Let \(n_p \geq n_q\). For each agent, there exists a local dynamic compensator such that the resulting system is represented by (10) and (11) where non-singular \(M \in \mathbb{R}^{p \times p}\) and \(R \in \mathbb{R}^{p \times p(n_u - 1)}\) are selected arbitrarily while \(A, B,\) and \(C\) are given by (6).

**PROOF.** See Appendix B.

Lemma 2 shows that a network of non-identical agents under disturbances can be converted to a network of agents which are partially identical. As \(H_i\) is Hurwitz stable, \(\tilde{x}_i\) and \(\rho_i\) have the same nature as \(\tilde{w}_i\), and one can redefine external disturbances as \(\tilde{w}_i = \text{col}\{\tilde{w}_i, \tilde{x}_i\}\). Hence, the model (10) is recast as (5) where \(E_i = [E_{d,i}, BW_i]\). Redefining disturbances changes the \(H_\infty\) norm; however, as the \(H_\infty\) norm of the system (11) is constant, it does not affect the solvability of the problem and \(H_\infty\) almost synchronization can be achieved for any given \(\gamma > 0\) by an appropriate choice of \(\epsilon\).

**Remark 2** In Lemma 2, \(R\) and \(M\) can be chosen arbitrarily to satisfy the designer’s wish. Actually, Lemma 2 states that agents can be shaped into the dynamics of any invertible system with no invariant zeros and of uniform rank \(n_q\). Such a system may be transformed into the form of (B.2) in accordance with Lemma 4 presented in Appendix B. The spectrum \(\lambda(A + BR)\) characterizes the time-response specifications of the selected system. Thus, Lemma 2 provides a wide range of choices for the control engineer.

### 4.5 Design Scheme

Here, the goal is to summarize the section and present a road map to design the protocol for multi-agent systems. Fig. 3 would help the reader to quickly grasp the vivid picture of the paper and realize how a network of non-identical agents is synchronized. Given a multi-agent system of the form (1), the design methodology is two-stage.

First, we represent the multi-agent system with non-identical members as a network of almost identical agents. As explained in Section 4.4, using the local measurement (1c), a local output feedback along with pre-compensators is designed for each agent so as to force agents to imitate a desired dynamic behavior even though the disturbance matrices are different from one agent to another. Shaping of agents into the desired structure, which is essentially a chain of integrators, is the key element for solvability of the \(H_\infty\) almost synchronization problem.

Next, the parameterized control law (7) is developed for the obtained almost identical multi-agent system (5), which ensures that synchronization can be accomplished with any arbitrarily small \(H_\infty\) performance if \(L \in G_\beta\) for any \(\beta > 0\).
5 $\mathcal{H}_\infty$ Almost Regulation of Output Consensus

Section 4 concerns the leaderless synchronization where an agreement amongst agents is paramount. Put clearly, the proposed protocol prescribes no restriction on the consensus trajectories. However, in many applications, it is desirable to regulate the output of each individual agent, $y_i$, to a particular trajectory while synchronizing them. The section tackles this problem.

5.1 Problem Formulation

Consider a reference trajectory $y_0 \in \mathbb{R}^p$ which is defined as the output of an exosystem in the form

$$\Sigma_0: \begin{cases} \dot{x}_0 = A_0 x_0 \\ y_0 = C_0 x_0 \end{cases}$$

where $x_0 \in \mathbb{R}^{n_0}$ and $(A_0, C_0)$ is observable. Recalling the definition of the mutual disagreement, consider $e_{i,0} = y_i - y_0, \forall i \in S$ as the regulation error for each agent. The problem is posed as “$\mathcal{H}_\infty$ almost regulation of output consensus” which is clearly expressed in Definition 4. Let

$$c_0 = \text{col}\{e_{i,0}\}, \quad c_0 = T_{w_c}(s)w$$

Definition 4 Given a set of networks $\mathcal{G}$, the problem of $\mathcal{H}_\infty$ almost regulation of output consensus with respect to a reference $y_0$ evolved from (12) for the multi-agent system (1) with the communication topology $\mathcal{L} \in \mathcal{G}$ is to find, if possible, a linear time-invariant dynamic protocol such that for any $\gamma > 0$, the closed-loop transfer function from $w$ to $c_0$ satisfies

$$\|T_{w_c}(s)\|_\infty < \gamma$$

Assumption 2 Every node of the network graph $\mathcal{L}$ is a member of a directed tree with the root contained in the “root set” $\pi \subset S$.

Note that if the network graph is one connected component containing a spanning tree, the set $\pi$ may only own one node which is the root of a spanning tree.

A certain subset of agents must know how far their outputs are from the reference $y_0$; otherwise, regulation is not possible. The set $\pi$ contains those agents which receive $e_{i,0}$ via the network. It implies that the network measurement (1d) is altered to

$$\tilde{\zeta}_i = \sum_{j=1}^N a_{ij} (y_i - y_j) + \psi_i (y_i - y_0)$$

where

$$\begin{cases} \psi_i > 0, \text{ if } i \in \pi; \\ \psi_i = 0, \text{ otherwise.} \end{cases}$$

Now, the exosystem is regarded as a new node, indexed by 0, and added to the network graph. The resulting graph is called the augmented network graph, denoted $\tilde{\mathcal{L}}$. If $\psi = \text{col}\{\psi_i\}$ and $\Psi = \text{diag}\{\psi_i\}$ for $i \in S$, the augmented Laplacian, associated with $\tilde{\mathcal{L}}$, is represented by $\tilde{L} = [\tilde{L}_{ij}]$ and is given by

$$\tilde{L} = \begin{bmatrix} 0 & 0 \\ -\psi L + \Psi \end{bmatrix}$$

Assumption 2 ensures that the augmented graph $\tilde{L}$ has a directed spanning tree (see Grip et al., 2012).

Definition 5 For given $\beta > 0$ and integer $N_0 \geq 1, \mathcal{G}_{\beta,\pi}$ is the set of network graphs composed of $N$ nodes where $N \leq N_0$ such that all graphs $\mathcal{L} \in \mathcal{G}_{\beta,\pi}$ satisfy Assumption 2 with one identical root set $\pi$ and the eigenvalues of their augmented graphs $\tilde{\mathcal{L}}$, denoted $\tilde{\lambda}_0, \tilde{\lambda}_1, \cdots, \tilde{\lambda}_N$, satisfy

$$\text{Re}\{\tilde{\lambda}_i\} > \beta, \quad \tilde{\lambda}_i \neq 0$$

One may recast (13) in terms of the augmented Laplacian as:

$$\tilde{\zeta}_i = \sum_{j=0}^N \tilde{L}_{ij} y_j$$

Likewise, the additional information (1e) is adapted according to the augmented network topology and is given by

$$\hat{\zeta}_i = \sum_{j=0}^N \tilde{L}_{ij} \eta_j$$

where $\eta_j \in \mathbb{R}^p$ will be specified later. As agent 0, which is the exosystem, receives no information from the network, $\zeta_0 = \hat{\zeta}_0 = 0$. Let us denote:

$$\tilde{\zeta} = \text{col}\{\tilde{\zeta}_i\}, \quad \tilde{\zeta} = T_{w_{\tilde{\zeta}}}(s)w$$

for $i \in S$. Note that $\tilde{\zeta} = 0$ means simultaneous achievement of synchronization and output regulation; i.e.

$$y_1 = \cdots = y_N = y_0$$

The problem defined by Definition 4 can be equivalently stated and solved by choosing $\tilde{\zeta}$ as the controlled output. Since Assumption 2 holds, the augmented graph has a directed spanning tree and it follows from Lemma 3 that $\|T_{w_{\tilde{\zeta}}(s)}\|_\infty < \epsilon \gamma$ for a given $\gamma > 0$ is equivalent to $\|T_{w_{\tilde{\zeta}}(s)}\|_\infty < \epsilon \gamma'$ for some $\gamma' > 0$.

5.2 Result 2: $\mathcal{H}_\infty$ Almost Regulation of Output Consensus

Theorem 2 Under Assumption 1 and for the set $\mathcal{G}_{\beta,\pi}$, the problem of $\mathcal{H}_\infty$ almost regulation of output consen-
sus is solvable; specifically, there exists a family of linear time-invariant protocols, parameterized in terms of a tuning parameter $\epsilon \in (0, 1]$, of the form

\[
\begin{align*}
\dot{X}_i &= A_i(c)X_i + B_i(c)\col\{\zeta_i, \tilde{\zeta}_i, y_m, i\} \\
u_i &= C_i(c)X_i + D_i(c)\col\{\zeta_i, \tilde{\zeta}_i, y_m, i\}
\end{align*}
\]

(17a)

(17b)

where $X_i \in \mathbb{R}^{n_i}$ and $i \in S$ such that

(i) for any $\beta > 0$, there exists an $\epsilon^* \in (0, 1]$ such that if $\epsilon \in \left(0, \epsilon^*\right]$, output regulation to the reference $y_0$ is accomplished in the absence of disturbance; i.e.,

\[
\forall \epsilon \in \left(0, \epsilon^*_i\right], \text{ when } w = 0 \quad \epsilon_{i,0} = y_i - y_0 \rightarrow 0, \forall i \in S \quad \text{as } t \rightarrow \infty
\]

(ii) for any given $\gamma > 0$, there exists an $\epsilon^*_i \in \left(0, \epsilon^*_i\right]$, such that if $\epsilon \in \left(0, \epsilon^*_i\right]$, the closed-loop transfer function from $w$ to $c_i$ satisfies

\[
\|T_{we_i}(s)\|_\infty < \gamma
\]

\[\Box\]

The proof is given in a constructive way in the following subsections. We move along the same threads of thought as in Section 4; we first make the augmented network behave as a multi-agent system with almost identical members; then, the protocol is designed for the resulting system.

We point out that Theorem 2 implies in the absence of disturbance, if $\epsilon_{i,0} = 0$ holds, then $\epsilon_{i,j} = 0$ holds; i.e., synchronization and output regulation are achieved simultaneously. Also, it renders $\|T_{we_i}(s)\|_\infty < \gamma$ when $\|T_{we_i}(s)\|_\infty < \gamma$.

The problem of regulation of output consensus is extremely straightforward using the proposed method as a result of shaping and because the pair $(A, B)$ is common between the exosystem and the agent. Therefore, the regulator equation is trivially solved.

### 5.3 Almost Identical Representation for Augmented Network

Let $n_q \geq n_{q_0}$ where $n_{q_0}$ is as Definition 3 for the augmented network. It can be demonstrated that, for any exosystem (12) satisfying observability condition, a series of internal state manipulations and a matrix $B_0 \in \mathbb{R}^{n_q \times p}$ always exist such that the triple $(A_0, B_0, C_0)$ is invertible and of the uniform rank $n_q$ with no invariant zeros. According to Lemma 4, the exosystem can be further represented in the following form:

\[
\Sigma_0 : \begin{cases}
\dot{x}_0 = Ax_0 + B(u_0 + R_0x_0) \\
y_0 = Cx_0
\end{cases}
\]

(18a)

(18b)

where $x_0 \in \mathbb{R}^{p_{n_q}}, M_0M_0^t = u_0 \in \mathbb{R}^p, A, B$ and $C$ are as (6) and $R_0 \in \mathbb{R}^{p \times p}$ while $M_0 \in \mathbb{R}^{p \times p}$ is nonsingular. Since the exosystem is autonomous and we have no control over it, $u_0$ must be zero.

In accordance with Lemma 2, there exist local dynamic pre-compensators that shape agents into (10) and (11) for the chosen $R_0$ and $M_0$. Consequently, a network of almost identical agents (5) emerges.

### 5.4 Dynamic Protocol

Considering the obtained multi-agent system (5) with the particular choice of $R_0$ and $M_0$, the protocol is proposed based on (7) in which $\zeta_i$ and $\tilde{\zeta}_i$ are replaced with $\zeta_i$ and $\tilde{\zeta}_i$, respectively. Therefore, the protocol is given by

\[
\begin{align*}
\dot{x}_i &= Ax_i + Bu_i + R_0\hat{x}_i - \epsilon^{-1}K(\tilde{\zeta}_i - \zeta_i) \\
u_i &= \epsilon^{-\alpha_S}FS\hat{x}_i
\end{align*}
\]

(19a)

(19b)

for $i \in \tilde{S} \triangleq \{0\} \cup S$. In (19), $\hat{x}_i \in \mathbb{R}^{p_{n_q}}$ and $\epsilon \in (0, 1]$ is the tuning parameter. Matrix $S$ is as (8) and $K$ is partitioned as (9). $F$, $K$ and $K_1$ are chosen similar to the procedure presented in Subsection 4.3. $\tilde{\zeta}_i$ is given by (16) where $\eta_j = C\hat{x}_j$. Notice that setting $\hat{x}_0(0) = 0$ leads to $\hat{x}_0(t) = u_0(t) = 0$ for $t \geq 0$.

### 6 $\mathcal{H}_\infty$ Almost Formation

The proposed synchronization method lends itself to formation control. In formation control, the objective is to maintain the relative outputs among agents as desired. In the presence of external disturbances, the problem of formation with the desired $\mathcal{H}_\infty$ performance is posed.

Let formation be defined in terms of a set of formation vectors $\mathcal{F} \triangleq \{f_1, \ldots, f_N\}, f_i \in \mathbb{R}^p$. Then, we define the output error of formation as

\[
y_{f,i} = y_i - f_i
\]

for $i \in S$. The mutual disagreement is then cast as

\[
\tilde{e}_{i,j} = y_{f,i} - y_{f,j}, \quad \forall i, j \in S, \quad i > j
\]

The transfer function $T_{we,f}(s)$ is defined as follows:

\[
\mathbf{e}_f \triangleq \col\{\tilde{e}_{i,j}\}, \quad \mathbf{e}_f = T_{we,f}(s)\mathbf{w}
\]

**Definition 6** Given a set of networks $\mathcal{G}$, the problem of $\mathcal{H}_\infty$ almost formation with respect to a formation set $\mathcal{F}$ for the multi-agent system (1) with the communication topology $\mathcal{L} \in \mathcal{G}$ is to find, if possible, a linear time-invariant dynamic protocol such that for any $\gamma > 0$, the closed-loop transfer function from $\mathbf{w}$ to $\mathbf{e}_f$ satisfies

\[
\|T_{we,f}(s)\|_\infty < \gamma
\]
Formation is possible when agents exchange output errors of formation; i.e. $y_{f,i}$’s. Therefore, the network information (1e) is to be modified to

$$\zeta_{f,i} = \zeta_i - \sum_{j=1}^{N} l_{ij} y_{f,j} = \sum_{j=1}^{N} l_{ij} y_{f,j}$$

The formation controller relies extensively on the fact that shaping is viable to any invertible system of uniform rank $n_q \geq n_{q0}$ which has no invariant zeros. Thus, Lemma 2 guarantees existence of local feedback laws to shape the system into the desired structure as (5).

In view of a multi-agent system (5), we propose the parameterized dynamic protocol

$$\begin{align*}
\dot{x}_i &= A\hat{x}_i + B(u_i + R\hat{x}_i) - \epsilon^{-1} K(\zeta_{f,i} - \hat{\zeta}_i) \\
u_i &= \epsilon^{-n_q} FS\hat{x}_i - R_1 f_i
\end{align*}$$

(21)

for $i \in S$. In (21), $\hat{x}_i \in \mathbb{R}^{n_q}$ and $\epsilon \in (0,1]$ is the tuning parameter. Matrix $S$ is as (8) and $K$ is partitioned as (9). $F$, $K$ and $K_1$ are chosen similar to the procedure presented in Subsection 4.3. The quantity $\zeta_{f,i}$ is found using (20). Considering $\eta_i = C\hat{x}_i$, $\hat{\zeta}_i$ is given by (1e). Theorem 3 states the result formally.

**Theorem 3** Under Assumption 1 and for the set $G_\beta$, the problem of $\mathcal{H}_\infty$ almost formation is solvable; specifically, there exists a family of linear time-invariant dynamic protocols, parameterized in terms of a tuning parameter $\epsilon \in (0,1]$, of the form

$$\begin{align*}
\dot{x}_i &= A_{\epsilon}(x) \chi_i + B_{\epsilon}(x) \text{col} \{\zeta_{f,i}, \hat{\zeta}_i, y_m,i\} \\
\nu_i &= C_{\epsilon}(x) \chi_i + D_{\epsilon}(x) \text{col} \{\zeta_{f,i}, \hat{\zeta}_i, y_m,i\} + F_{\epsilon}(f_i)
\end{align*}$$

(22)

where $\chi_i \in \mathbb{R}^n$ and $i \in S$ such that

(i) for any $\beta > 0$, there exists an $\epsilon_1^* \in (0,1]$ such that if $\epsilon \in (0,\epsilon_1^*]$ the desired formation is attained in the absence of disturbance; i.e. $\forall \epsilon \in (0,\epsilon_1^*] \text{ when } w = 0$

$$\bar{e}_{i,j} \to 0 \text{ as } t \to \infty$$

(ii) for any given $\gamma > 0$, there exists an $\epsilon_2^* \in (0,\epsilon_1^*]$ such that if $\epsilon \in (0,\epsilon_2^*]$, the closed-loop transfer function from $w$ to $e_f$ satisfies

$$\|T_{w,e_f}(s)\|_\infty \leq \gamma$$

**Proof.** See Appendix C.

**Remark 3** It is worth noting that the given protocol imposes no restrictions on the agreement trajectories; that is, although agents establish the desired configuration, it is not clear where the whole system heads to. The formation control can be combined with the regulation problem, arising the problem of $\mathcal{H}_\infty$ almost formation with regulation of output consensus” which is illustrated by simulation.

7 Illustrative Example

The result is illustrated for a network consisting of $N = 4$ right-invertible agents with $p = 1$. The interconnection topology of the network is given by the digraph displayed in Fig. 4a. The corresponding Laplacian is

$$L = \begin{bmatrix}
6 & -2 & -4 & 0 \\
0 & 3 & 0 & -3 \\
-5 & 0 & 0 & 5
\end{bmatrix}$$

The models of agents are given in Appendix E. Disturbances are chosen $w_1 = \sin(t), w_2 = 1, w_4 = \sin(2t)$, and $\|w_3\| \leq 5$ which is a uniform random number. The order of the infinite zeros of agent 1 to 4 are respectively 3, 2, 1, and 2. Thus, agent 1 has the largest order of infinite zeros; i.e. $n_{q0} = 3$; we choose $n_q = n_{q0}$.

7.1 Network Shaping

The first step is to design a local output feedback for each agent to have an almost identical representation for the network. Agents 1 and 2 are invertible, but agents 3 and 4 need to be squared down. The pre-compensator $\Sigma_{1,3}$ squares down agent 3 and locates the additional invariant zeros at $\{-2,-2,-2,-2,-1\}$. Placing the additional infinite zeros at $\{-2,-2,-1\}$, the interconnection of $\Sigma_{1,4}$ and agent 4 is invertible. The dynamic equations of the compensators are given in Appendix F.1.

Together with the pre-compensators, all the agents are invertible and single-input single-output. Now, we make all of them of the same relative degree $n_q$. Agents 2, 3, and 4 are compensated by $\Sigma_{2,2}, \Sigma_{2,3}$ and $\Sigma_{2,4}$, represented in Appendix F.2.

![Fig. 4](image-url)
After pre-compensation agent 1-4 are of orders 3, 3, 8 and 7, respectively. The third step to achieve an almost identical representation is to design output feedbacks for the pre-compensated agents, which requires linear observers. The required information is presented in Appendix F.3.

7.2 $H_\infty$ Almost Synchronization

The appropriately shaped agents are now considered and the control law is produced according to (7). We choose $M = 1$ and $R = [0, -1, 0]$ to have $\lambda(A + BR) = \{0, \pm i\}$ for the synchronization problem. We select $K_1 = -1$. $F$ and $K$ are selected such that $\lambda(A + BF) = \{-3, -4, -5\}$ and $\lambda(\bar{A}_z) = \{-2, -3\}$. Fig. 5 shows the result for $\epsilon = 0.01$ and $\epsilon = 0.05$. The smaller the value of $\epsilon$ is, the smaller $\xi_i$ is, the more $u$ is rejected from the mutual disagreements.

7.3 $H_\infty$ Almost Regulation of Output Consensus

The consensus trajectory for the multi-agent system is desired to be $\sin(\omega_0 t)$. The exo-system (12) is then given by

$$\dot{x}_0 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} x_0 + \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{z}_i \quad (23)$$

with $x_0(0) = [2, 0]^T$. Thus, $n_q = n_{q0} = 3$ is preserved, but the exo-system has to be shaped as desired. If $B_0 = [0, 1]^T$ is chosen, the resulting system is invertible and one may design a rank-equalizing pre-compensator to change the relative degree to $n_q$. Transforming the system using the observability matrix renders the exo-system as (18) with

$$R_0 = \begin{bmatrix} 0, -\omega_0^2, 0 \end{bmatrix}, \quad M_0 = 1$$

and $x_0 = [2, 0, -2\omega_0^2]^T$. Agent 1 is the root of one spanning tree in the network graph, and we link agent 0 with agent 1 with weight $\nu_1 = 1$ as depicted in Fig. 4b. In fact, we choose $\pi = \{1\}$. Let $\omega_0 = 0.5$. Fig. 6 displays the results, where the outputs of all agents are plotted, for $\epsilon = 0.01$ and $\epsilon = 0.05$. As expected, smaller $\epsilon$ leads to a better disturbance rejection. To discern the contrast between the leaderless synchronization and the regulation of output consensus, the consensus trajectories for each case are plotted in Fig. 7. It is established from Fig. 7 that the consensus trajectory for the leaderless synchronization is unbounded although $\lambda(A + BR) = \{0, \pm i\}$ implies that the response of the system would be oscillatory; however, the network is driven by disturbance. By contrast, the consensus trajectory in the regulation of output consensus is close to what is desired.
H∞ almost formation with output regulation - ϵ = 0.01
yi − y0, i ∈ S

Time (sec)

H∞ almost formation with output ... − y0
y3 − y0
y4 − y0
y2 − y0
y4
y2
0 5 10 15 20
0 5 10 15 20
− 10
− 5
0
5
10
− 15
− 10
− 5
0
5
10
15

Fig. 8. H∞ almost formation control with output regulation; the upper plot shows y_i, i ∈ S and the lower plot depicts the regulation error y_i − y_0, i ∈ S.

7.4 H∞ Almost Formation Control with Regulation

This part builds up H∞ almost regulation of output consensus into formation control. To be specific, agents have to form a desired configuration while they are forced to follow the reference y_0. This can be portrayed as virtual reference formation control. Simulation is carried out for the same network as before. The formation set is selected as

S_f = \{10, 5, -5, -10\}

Thus, the objective is to have

y_1 - 10 = y_2 - 5 = y_3 + 5 = y_4 + 10 = y_0 \pm \sin(0.5t)

as time approaches to infinity. The result is given in Fig. 8.

References


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A Proof of Lemma 1

Before embarking on the proof, we would like to stress that the problem of \(H_\infty\) almost synchronization can be stated equivalently by choosing \(\zeta\) as the controlled output and designing a protocol such that \(\|T_{\zeta}(s)\|_\infty < \gamma'\) for some \(\gamma' > 0\) where

\[\zeta = T_{\zeta}(s)w\]

In this case, the objective is to approximately decouple \(\zeta\) from \(w\) to the desired level in the sense of the \(H_\infty\) norm of \(T_{\zeta}\). We shall show in Lemma 3 that these two formulations are equivalent.

To prove Lemma 1, we first write the closed-loop equations after suitable state transformations; second, the unobservable modes are removed and the order of the system from \(w\) to \(\zeta\) is reduced. Then, the \(H_\infty\) norm of the transfer function based on the reduced order system is established. Finally, we show that the \(H_\infty\) norm of the transfer function from \(w\) to \(e_{i,j}\) can be made arbitrarily small.

A.1 Closed-loop Equations

Let \(x^*_i \triangleq x_i - \hat{x}_i\) be the observation error for agent \(i\); then, in view of (5) and (7), the closed-loop equations for agent \(i\) can be written as

\[\dot{x}_i^* = Ax_i + BRx_i + e^{-\eta w}BFS(x_i - x^*_i) + E_iw_i\]

\[\dot{z}_i = A_zz_i + R_sz_i + \epsilon^{-1}C^TK_iC \sum_{j=1}^{N} l_{ij}z_j + SE_iw_i\]

Let \(R = [R_1, \bar{R}]\) where \(R_1 \in \mathbb{R}^{p \times p}\) and

\[\bar{R} = \begin{bmatrix} R_2 & \cdots & R_n \end{bmatrix}, \quad \text{where} \quad R_i \in \mathbb{R}^{p \times p}\]

Consider the following state transformations

\[e_i = Sx_i, \quad z_i = Sx_i^*, \quad S \triangleq \begin{bmatrix} I_p & 0 \\ -\epsilon K & \epsilon I_{p(n-1)} \end{bmatrix}\]

The closed-loop equations can be then recast as:

\[\dot{e}_i = \epsilon^{-1}(A + BF)e_i + R_1e_i - \epsilon^{-1}BFS^{-1}z_i + SE_iw_i\]

\[\dot{z}_i = A_zz_i + R_sz_i + \epsilon^{-1}C^TK_iC \sum_{j=1}^{N} l_{ij}z_j + SE_iw_i\]

in which

\[R_e = B \begin{bmatrix} \epsilon^{n-1}R_1, \epsilon^{n-2}R_2, \cdots, R_n \end{bmatrix}\]

\[R_z = \epsilon B \begin{bmatrix} R_1 + \bar{R}K, \epsilon^{-1}\bar{R} \end{bmatrix}, \quad A_z = \begin{bmatrix} C_1K & \epsilon^{-1}C_1 \\ \epsilon K'K & K' \end{bmatrix}\]

where \(K' = A_1 - \bar{K}C_1\). We split \(z_i\) into \(z_{1,i} = Cz_i \in \mathbb{R}^p\) and \(z_{2,i}\) such that \(z_i = \text{col} \{z_{1,i}, z_{2,i}\}\). Let

\[F = \begin{bmatrix} F_1 & F_2 & \cdots & F_{n_q} \end{bmatrix}\]

\[K = \begin{bmatrix} \bar{K}_1^T & \bar{K}_2^T & \cdots & \bar{K}_{n_q-1}^T \end{bmatrix}^T\]

where \(F_i \in \mathbb{R}^{p \times p}, \bar{K}_i \in \mathbb{R}^{p \times p}\). We represent

\[FSS^{-1} = \begin{bmatrix} F_1^* & F_2^* \end{bmatrix}\]

\[F_1^* = F_1 + \sum_{s=1}^{n_q-1} \epsilon^s F_{s+1}\bar{K}_s\]

\[F_2^* = \begin{bmatrix} F_2 & \epsilon F_3 & \cdots & \epsilon^{n_q-3} F_{n_q-1} & \epsilon^{n_q-2} F_{n_q} \end{bmatrix}\]
Defining \( \tilde{E}_{2,i} \triangleq E_{2,i} - KE_{1,i} \) and \( \tilde{E}_2 \triangleq \tilde{A}_1 K + B_1 R_1 \), one may show the closed-loop equations as:

\[
\begin{align*}
\dot{e}_i &= (A + BF) e_i + \epsilon R e_i - BF^* z_{1,i} - BF z_{2,i} + \epsilon SE_i w_i, \\
\dot{z}_{1,i} &= \epsilon C_1 \tilde{z}_{1,i} + C_1 z_{2,i} + K \sum_{j=1}^{N} l_{ij} z_{1,j} + \epsilon E_{1,i} w_i, \\
\dot{z}_{2,i} &= \epsilon \tilde{E}_2 z_{1,i} + \tilde{A}_2 z_{2,i} + \epsilon \tilde{E}_{2,i} w_i.
\end{align*}
\]

For \( i \in \mathcal{S} \), consider the following notations:

\[
G = \text{diag}\{SE_i\}, \quad \tilde{G}_1 = \text{diag}\{E_{1,i}\}, \quad \tilde{G}_2 = \text{diag}\{\tilde{E}_{2,i}\}, \quad e = \text{col}\{e_i\}, \quad z_1 = \text{col}\{z_{1,i}\}, \quad z_2 = \text{col}\{z_{2,i}\}.
\]

Considering the network Laplacian \( L = [l_{ij}] \) for \( i, j \in \mathcal{S} \), the closed-loop equations for the network are given by

\[
\begin{align*}
\dot{e} &= (L_N \otimes (A + BF)) e + (L_N \otimes \epsilon R e) e \\
&- (L_N \otimes BF^* z_{1}) z_1 - (L_N \otimes BF z_{2}) z_2 + \epsilon \tilde{G}_e w, \\
\dot{z}_1 &= (L_N \otimes \epsilon C_1 \tilde{K} + L \otimes K) z_1 \\
&+ (L_N \otimes C_1) z_2 + \epsilon \tilde{G}_1 w, \\
\dot{z}_2 &= (L_N \otimes \epsilon \tilde{E}_2) z_1 + (L_N \otimes \tilde{A}_2) z_2 + \epsilon \tilde{G}_2 w, \\
\zeta &= (L \otimes C) e.
\end{align*}
\]

**A.2 Reduced-order Dynamics**

One right eigenvector of \( L \) is \( 1 \in \mathbb{R}^N \). Let \( 1_L \) represent its left eigenvector. Suppose the Jordan form of \( L \) is obtained using the matrix \( U \) which is chosen as

\[
U = \begin{bmatrix} \bar{U} & 1 \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} U_L^T \\ I_L \end{bmatrix}
\]

It implies that \( U_L^T \bar{U} = L_{N-1} \), \( U_L^T 1 = 1_L U = 0 \), and \( I_L^T 1 = 1 \). Thus, one can find

\[
U^{-1} L U = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad L U = \begin{bmatrix} \bar{L} & 0 \end{bmatrix}
\]

where \( \bar{L} = \bar{U} \Delta \). Since \( \mathcal{L} \in \mathcal{G}_\beta \), all eigenvalues of \( \Delta \) lies in \( \mathbb{C}^+ \). The following state transformations are introduced:

\[
\begin{align*}
\bar{e} &= (U^{-1} \otimes I_{pm_1}) e, \\
\bar{z}_1 &= (U^{-1} \otimes I_{p}) z_1, \\
\bar{z}_2 &= (U^{-1} \otimes I_{p}) z_2.
\end{align*}
\]

where \( \bar{p} = p(n_q - 1) \). Denoting \( N = N - 1 \), the states \( \bar{e}, \bar{z}_1 \), and \( \bar{z}_2 \) are vectors of the dimensions \( \bar{N} \times pm_1, \bar{N} \times p \) and \( \bar{N} \times \bar{p} \), respectively. Obviously, the states \( e^*, z_1^* \) and \( z_2^* \) are vectors of the dimensions \( pm_1, p \) and \( \bar{p} \), respectively. Consider the following notations:

\[
\begin{align*}
\bar{G}_e &= (U_L^T \otimes I_{pm_1}) G, \\
\bar{G}_1 &= (U_L^T \otimes I_{p}) \tilde{G}_1, \\
\bar{G}_2 &= (U_L^T \otimes I_{p}) \tilde{G}_2, \\
\bar{e} &= (L \otimes C) \bar{e}.
\end{align*}
\]

As a result, the system dynamics is divided into two subsystems. One subsystem is of order \( 2pm_1 \bar{N} \) and is given by

\[
\begin{align*}
\epsilon \bar{e} &= (L_\bar{N} \otimes (A + BF)) \bar{e} + (L_\bar{N} \otimes \epsilon R e) \bar{e} \\
&- (L_\bar{N} \otimes BF^* z_1^*) z_1 - (L_\bar{N} \otimes BF z_2^*) z_2 + \epsilon \bar{G}_e w, \\
\bar{e}_1 &= (L_\bar{N} \otimes \epsilon C_1 \bar{K} + \bar{L} \otimes K) z_1 \\
&+ (L_\bar{N} \otimes C_1) z_2 + \epsilon \bar{G}_1 w, \\
\bar{e}_2 &= (L_\bar{N} \otimes \epsilon \bar{E}_2) z_1 + (L_\bar{N} \otimes \bar{A}_2) z_2 + \epsilon \bar{G}_2 w, \\
\zeta &= (L \otimes C) \bar{e}.
\end{align*}
\]

It follows from (A.5d) that \( \zeta \) is only affected by \( \bar{e} \) since the chosen output for the system only captures disagreement between agents and it depends only on \( \epsilon_{i,j} = \bar{y}_i - \bar{y}_j \). The other subsystem is of order \( 2pm_1 \bar{N} \) and is given by

\[
\begin{align*}
\epsilon \bar{e}^* &= (A + BF) \epsilon^* + \epsilon R e^* \\
&- BF^* z_1^* - BF z_2^* + \epsilon \bar{G}_e^* w, \\
\bar{e}_1^* &= \epsilon C_1 \bar{K} z_1^* + C_1 z_2^* + \epsilon \bar{G}_1^* w, \\
\bar{e}_2^* &= \epsilon \bar{E}_2 z_1^* + \bar{A}_2 z_2^* + \epsilon \bar{G}_2^* w.
\end{align*}
\]

When all agents have reached an agreement, \( \zeta \) is zero which does not impose any constraints on the dynamic system (A.6). In fact, (A.6) determines the consensus trajectories when \( \zeta = 0 \). It enunciates the fact that the consensus trajectories may be unbounded.

It is then inferred that the objective in the network synchronization is to design a protocol such that the reduced-order dynamics (A.5) vanishes in time. Eventually, the \( \mathcal{H}_\infty \) almost synchronization problem for the network of agents is converted to the \( \mathcal{H}_\infty \) control of the reduced-order dynamics.

**A.3 \( \mathcal{H}_\infty \) Analysis**

We prove the theorem for \( \|T_{w_\infty}\|_\infty \) as the transfer function of the reduced order system (A.5). According to (A.5d), \( \| \zeta \| = \rho_\zeta \| \bar{e} \| \) for some \( \rho_\zeta \) independent of \( \epsilon \). As \( A + BF \) is Hurwitz stable, there exists \( P_{\epsilon} = P_{\epsilon}^T > 0 \) which solves the following Lyapunov function:

\[
(T_{N} \otimes (A + BF))^T P_{\epsilon} + P_{\epsilon} (T_{N} \otimes (A + BF)) = -2I
\]
Choose the positive definite function \(W_c = c \hat{e}^T P_c \hat{e}\) and differentiate it along the trajectories of (A.5a). Denote
\[
\begin{align*}
    s_0 &= \|P_c (I_N \otimes R_e)\| \\
    s_1 &= \|P_c (I_N \otimes B F^*_e)\| \\
    \rho_1 &= \|P_c \hat{G}_e\| \\
    s_2 &= \|P_c (I_N \otimes B F^*_e)\|
\end{align*}
\]
Notice \(\epsilon s_0 = O(\epsilon)\) and \(s_1\) and \(s_2\) are bounded. For sufficiently small \(\epsilon\), we obtain
\[
(1 - \epsilon s_0) > \frac{1}{2} \quad (A.7)
\]
Then, one can find the upper bound for \(\hat{W}_c\) as
\[
\hat{W}_c \leq -\|\hat{e}\|^2 + 2\rho_2 \sqrt{\|z_1\|^2 + \|z_2\|^2} \|\hat{e}\| + 2\epsilon \rho_1 \|\hat{e}\| \|W\| \quad (A.8)
\]
where \(\rho_2 = \sqrt{2} \max\{s_1, s_2\}\). To find the upper bound (A.8), we have made use of the fact that \(x + y \leq \sqrt{2} \sqrt{x^2 + y^2}\). Since \(K_1 < 0\) and \(\lambda(\Delta) \in \mathbb{C}^+, \Delta \otimes K_1\) is Hurwitz stable. Thus, the Lyapunov equation
\[
(\Delta \otimes K_1)^\top P_1 + P_1 (\Delta \otimes K_1) = -Q_1 \quad (A.9)
\]
has a unique solution \(P_1 > 0\). Recalling Definition 1, as \(\mathcal{L} \in \mathcal{G}_\beta, \Re\{\lambda_\ell(\Delta)\} > \beta\), for \(i = 1, \ldots, N\).

**Proposition 1** For any \(\beta > 0\), there exists a bounded \(P_1 > 0\) such that the Lyapunov equation (A.9) holds and
\[
\|Q_1\| > 4q \quad (A.10)
\]
where \(q = q(\beta)\).

**PROOF.** See Appendix D.

Proposition 1 states that for the set \(\mathcal{G}_\beta\), we can construct a block diagonal matrix \(P_1 > 0\), which is bounded; \(P_1\) solves the Lyapunov function (A.9) for \(Q_1 > 0\) such that \(\|Q_1\|\) is bounded from below by a function of \(\beta > 0\). Choose \(W_1 = q \hat{z}_1^2 P_1 \hat{z}_1\) and differentiate it in time. The upper bound for \(W_1\) is then given by
\[
\hat{W}_1 \leq -2q^2 \|\hat{z}_1\|^2 - 2q \|\hat{z}_1\|^2 (q - \epsilon s_3) + 2q \|\hat{z}_1\| \|\hat{z}_2\| + 2q \|\hat{z}_1\| \|\hat{z}_2\| \|W\|
\]
where \(\rho_3 = \|P_c \hat{G}_e\|\) and
\[
s_3 = \|P_c (I_N \otimes C_1 \hat{K})\| \quad s_4 = \|P_c (I_N \otimes C_1)\|
\]
Since \(\hat{A}_e\) is Hurwitz stable, the equation
\[
(I_N \otimes \hat{A}_e)^T P_2 + P_2 (I_N \otimes \hat{A}_e) = -(2q + q^{-1} s_4^2) \mathcal{I}
\]
has the unique solution \(P_2 = P_2^T > 0\). The derivative of \(W_2 = q \hat{z}_2^2 P_2 \hat{z}_2\) along the trajectories of (A.5c) is bounded by
\[
\hat{W}_2 \leq -(2q^2 + s_3^2) \|\hat{z}_2\|^2 + 2q \epsilon s_3 \|\hat{z}_2\| \|\hat{z}_1\| + 2q \epsilon \rho_4 \|\hat{z}_2\| \|W\|
\]
in which
\[
s_5 = \|P_2 (I_N \otimes \hat{G}_e)\| \quad \rho_4 = \|P_2 \hat{G}_e\|
\]
Consider \(W_o = W_1 + W_2\) and differentiate in time. One may find an upper bound for \(W_o\) as
\[
\hat{W}_o \leq -q^2 \|\hat{z}_1\|^2 - 2q \|\hat{z}_4\| \|\hat{z}_2\| + 2q \epsilon \rho_5 \sqrt{\|\hat{z}_1\|^2 + \|\hat{z}_2\|^2} \|W\|
\]
where \(\rho_5 = \sqrt{2} \max\{\rho_3, \rho_4\}\). Consequently, we select the following Lyapunov function for the system (A.5):
\[
V = (2 + q^2) W_c + (1 + (2 + q^2)^2 \rho_3^2 q^{-2}) W_o \quad (A.12)
\]
Differentiating \(V\) with respect to time yields
\[
\dot{V} \leq -\|\hat{e}\|^2 + 2(2 + q^2) \rho_2 \sqrt{\|\hat{z}_1\|^2 + \|\hat{z}_2\|^2} \|\hat{e}\| - (2 + q^2)^2 \rho_2^2 (\|\hat{z}_1\|^2 + \|\hat{z}_2\|^2) \|\hat{e}\|^2 - \rho_2^2 \|\hat{e}\|^2 - (Q^2 \|\hat{z}_1\|^2 + q^2 \|\hat{z}_2\|^2) \|\hat{e}\|^2 + 2\epsilon \rho_6 \sqrt{q^2 \|\hat{z}_1\|^2 + q^2 \|\hat{z}_2\|^2} \|\hat{e}\| \|W\|
\]
where \(\rho_6 = \sqrt{2} \max\{\rho_1, (2 + \rho_2^2), \rho_5 (1 + (2 + q^4)^2 \rho_2^2 q^{-2})\}\). The first two lines comprise a square, which is negative. Then, we arrive at
\[
\dot{V} \leq -q^2 \|\hat{e}\|^2 - (Q^2 \|\hat{z}_1\|^2 + q^2 \|\hat{z}_2\|^2) \|\hat{e}\|^2 + 2\epsilon \rho_6 \sqrt{q^2 \|\hat{z}_1\|^2 + q^2 \|\hat{z}_2\|^2} \|\hat{e}\| \|W\| \quad (A.13)
\]
Completing the square, it gives rise to
\[
\dot{V} + \|\hat{e}\|^2 (\epsilon \rho_6)^2 \|W\|^2 \leq 0 \quad (A.14)
\]
Therefore, it follows from Kalman-Yakubovich-Popov Lemma (Zhou and Doyle, 1998) that
\[
\|T_{wc}\| < \epsilon \rho_6
\]

and the contribution of $w$ to $\zeta$ vanishes as $\epsilon \to 0$. Note that (A.14) is obtained if $\epsilon \in (0, \epsilon_1^i)$ where $\epsilon_1^i$ is the largest $\epsilon$ which satisfies the conditions (A.7) and (A.11).

We would like to draw attention to the fact that $P_i$ is found for a set of networks, say $G_{ij}$, not for a given network $L \in G_{ij}$; accordingly, $s_3, s_4$, and $s_5$ are independent of one specific choice for the network graph.

Inequality (A.13) accentuates that $\bar{e}, \bar{z}_1, \bar{z}_2 \to 0$ exponentially fast and agreement is reached if $w = 0$ although it makes no conclusions about the agreement states ($e^*, z_1^*, z_2^*$). Thus, the agreement trajectories can be nonzero or even unbounded.

So far, we have shown that the proposed family of protocols can reject $w$ from $\zeta$ to the desired level. Lemma 3 demonstrates that (7) has a similar decoupling effect on $e$. We define

$$e_{i,j} = T_{w^2}(s)w$$

**Lemma 3** Given $\epsilon > 0$ and for $\gamma, \gamma' > 0$, the following statements are equivalent.

1. $\|T_{w^2}\|_\infty < \epsilon \gamma$
2. $\|T_{w^2}\|_\infty < \epsilon \gamma'$

**PROOF.** If (2) is given, (1) is deduced for some $\gamma > 0$ since $\zeta$ is the weighted sum of $e_{i,j}$’s. To show the other direction, by an appropriate choice of $A$ (which is Hurwitz stable) and $B$, the input-output representation of (A.5) is described by

$$\zeta = \epsilon (L \otimes C)(sI - A)^{-1}Bw$$  \hspace{1cm} (A.15)

We pick one agent arbitrarily. Let it be agent $N$. Due to zero row-sum property of the Laplacian, we have $\sum_{j=1}^{N} l_{ij}y_j = 0$. Thus, we recast the network measurement as

$$\zeta_i = \sum_{j=1}^{N} l_{ij}y_j - \sum_{j=1}^{N} l_{ij}y_N = \sum_{j=1}^{N} l_{ij}e_{j,N}$$

Let $\sigma_i = \zeta_i - \zeta_N$ and $\epsilon_N = \text{col}\{\epsilon_i, N\}$ for $i \in S_1$ where $S_1 = \{1, 2, \cdots, N\}$. Thus, one may find

$$\sigma_i = \sum_{j=1}^{N} l_{ij}'e_{j,N}$$

where $l_{ij}' = l_{ij} - l_{Nj}, j \in S$. Let $\bar{L} \triangleq [l_{ij}'] \in \mathbb{R}^{N \times N}$ be obtained by removing the last row of $L - 1 l_k^N$, where $l_k^N$ denotes the $k$th row of $L$. Let $L^* \triangleq [l_{ij}^*] \in \mathbb{R}^{N \times N}$ be the reduced Laplacian which is found by discarding the last column of $\bar{L}$. According to Yang et al. (2011a), $L^* > 0$. Therefore, defining $\sigma = \text{col}\{\sigma_i\}$, we obtain

$$\sigma = (L^* \otimes I_p)\epsilon_N$$

In view of (A.15),

$$\sigma = \epsilon (\bar{L} \otimes C)(sI - A)^{-1}Bw$$

Hence,

$$\epsilon_N = \epsilon (L^* \otimes I_p)^{-1}(\bar{L} \otimes C)(sI - A)^{-1}Bw$$

It shows that $T_{w_i}^* \epsilon$ depends on $\epsilon$, and $\|T_{w_i}^*\|_\infty < \epsilon \gamma'$ is attained for some $\gamma' > 0$.

**B Proof of Lemma 2**

Before starting the proof of Lemma 2, we recall the following result from Sannuti and Saberi (1987).

**Lemma 4** (Sannuti and Saberi 1987) Consider an invertible system which has no invariant zeros and is of uniform rank $n_q$ (i.e. all infinite zeros have the same order $n_q$). It is described as

$$\begin{cases}
\dot{x} = A\bar{x} + B\bar{u} \\
z = C\bar{x}
\end{cases}$$

where $\bar{x} \in \mathbb{R}^{n_{q}}$ and $z, \bar{u} \in \mathbb{R}^{p}$. There exist a nonsingular state transformation $\Gamma_0$ and a nonsingular input transformation $M$ such that $x = \Gamma_0 \bar{x}$ and $u = M\bar{u}$ transform the system into

$$\begin{cases}
\dot{x} = Ax + Bu + Rx \\
z = Cx
\end{cases}$$

where $A, B$, and $C$ are as (6). Also, $R \in \mathbb{R}^{p \times n_{q}}$. \hspace{1cm} \square

As stated in Section 4.4, the design procedure is three-step. Step 1: According to (Saberi and Sannuti, 1988), the squaring-down pre-compensator for the right-invertible system is given by

$$\begin{cases}
\dot{p}_{i,i} = A_{p_{i,i}}p_{i,i} + B_{p_{i,i}}u_{p_{i,i}} \\
u_{i} = C_{p_{i,i}}p_{i,i} + D_{p_{i,i}}u_{p_{i,i}}
\end{cases}$$

where $u_{p_{i,i}} \in \mathbb{R}^{p}$, and $A_{p_{i,i}}$ is Hurwitz. It makes agent $i$ invertible while adding new stable invariant zeros. Step 2: The rank-equalizing pre-compensator is designed based on (Saberi et al., 1990) and is given by

$$\begin{cases}
\dot{p}_{2,i} = A_{p_{2,i}}p_{2,i} + B_{p_{2,i}}u_{p_{2,i}} \\
u_{p_{i,i}} = C_{p_{2,i}}p_{2,i} + D_{p_{2,i}}u_{p_{2,i}}
\end{cases}$$
Defining $p_i = \text{col} \{ \tilde{x}_i, p_{1,i}, p_{2,i} \}$, the cascade interconnection of $\Sigma_{2,i}$, $\Sigma_{1,i}$ and agent $i$ can be shown by

$$
\begin{align}
\dot{p}_i &= \Lambda_{p,i}p_i + \Lambda_{u,i}u_{p_{2,i}} + \Lambda_{w,i} \tilde{w}_i \quad \text{(B.5a)} \\
y_i &= \begin{bmatrix} C_i & 0 \end{bmatrix} p_i \quad \text{(B.5b)}
\end{align}
$$

which is an invertible system, with uniform rank $n_q$.

Step 3: According to Samuti and Saberi (1987), there exist nonsingular state and input transformations such that $p_i = \Gamma_{2,i} \text{col} \{ x_{a,i}, x_i \}$ and $u_{p_{2,i}} = \Gamma_{2,i} u_{d,i}$ transform (B.5) into the special coordinate basis (s.c.b)

$$
\begin{align}
\dot{x}_{a,i} &= A_{a,i}x_{a,i} + L_{ad,i}y_i + E_{a,i} \tilde{w}_i \quad \text{(B.6a)} \\
\dot{x}_i &= Ax_i + B(u_{d,i} + R_{a,i}x_{a,i} + R_{d,i}x_i) + E_{d,i} \tilde{w}_i \quad \text{(B.6b)} \\
y_i &= Cx_i \quad \text{(B.6c)}
\end{align}
$$

where $x_{a,i} \in \mathbb{R}^{n_a}$ and $x_i \in \mathbb{R}^{n_v}$ represent the zero dynamics and the infinite-zero structure, respectively. $A$, $B$ and $C$ are given by (6). Obviously, one can find

$$
E_{o,i} \triangleq \begin{bmatrix} E_{a,i} \\ E_{d,i} \end{bmatrix} = \Gamma_{1,i}^{-1} A_{w,i}
$$

From now on, the goal is to make the system equations (B.6b) similar to (10). This is achieved by means of a feedback to decouple the zero dynamics from $x_i$ subsystem and add the required terms. Therefore, we need to estimate $p_i$. The measurement available for the system (B.5) is $y_{m,i}^* \triangleq C_{m,i}^* p_i = \text{diag} \{ C_{m,i}, I \} p_i$. Since $(A, C_{m,i})$ is detectable, the pair $(A_{p,i}, C_{m,i})$ is detectable, and one can design an observer to reconstruct $p_i$ by reading $y_{m,i}^*$ and $u_{p_{2,i}}$. Let the estimation error be $\tilde{x}_i \triangleq \text{col} \{ x_{a,i}, x_i \} - \text{col} \{ \hat{x}_{a,i}, \hat{x}_i \}$ where $\text{col} \{ \hat{x}_{a,i}, \hat{x}_i \}$ is the estimated signal. The dynamic equation of $\tilde{x}_i$ is given by (11a) where $H_i$ is Hurwitz stable. Now, we choose the following pre-feedback

$$
u_{d,i} = u_i - R_{a,i} \tilde{x}_{a,i} - R_{d,i} \hat{x}_i + R \hat{x}_i \quad \text{(B.7)}
$$

Considering $u_i = M \nu'_i$ and substituting (B.7) in (B.6b) give rise to (10) and (11). Thus, $W_i = [R_{a,i}, (R_{d,i} - R)]$. According to Fig. 2, $\nu'_i$ is the new input of agent $i$.

### C Proof of Theorem 3

It follows from Lemma 2 that there exists a dynamic compensator that makes agent $i \in S$ have the dynamics of (5) for an arbitrary $R$ and nonsingular $M$. $R \in \mathbb{R}^{p\times ps}$ is partitioned as $R = [R_1; \tilde{R}]$ where $R_1 \in \mathbb{R}^{p \times p}$. The vector $f_i \in \mathbb{R}^{ps}$ for $i \in S$ is formed as

$$
\tilde{f}_i = \begin{bmatrix} f_i \\ 0 \end{bmatrix} \Rightarrow f_i = Cf_i \quad \text{(C.1)}
$$

The state error of formation is then denoted $x_{f,i} = x_i - \tilde{f}_i$. In view of $A\tilde{f}_i = 0$, $R\tilde{f}_i = R_1 f_i$ and $\tilde{f}_i = 0$, (5) is recast as

$$
\begin{align}
\dot{x}_{f,i} &= Ax_{f,i} + B(u_{f,i} + Rx_{f,i}) + E_{i} w_i \quad \text{(C.2a)} \\
y_{f,i} &= Cx_{f,i} \quad \text{(C.2b)}
\end{align}
$$

where $u_{f,i} = u_i + R_1 f_i$. In compliance with Section 4.3, the observer-based protocol for (C.2) will take the following form

$$
\begin{align}
\dot{\hat{x}}_i &= A\hat{x}_i + B(u_{f,i} + R \hat{x}_i) - \epsilon^{-1} K \sum_{j=1}^{N} l_{ij} C \hat{x}_j \quad \text{(C.3a)} \\
u_{f,i} &= \epsilon^{-n_s} FS \hat{x}_i \quad \text{(C.3b)}
\end{align}
$$

in which $\hat{x}_j = x_{f,j} - \hat{x}_j$. It then yields closed-loop equations similar to those given in Section 4.3. The rest of the proof is akin to the proof of Lemma 1.

### D Proof of Proposition 1

We choose $P_1 = (-P \otimes K_1^{-1})$ in which $P > 0$ is diagonal as $P = \text{diag} \{ p_1, \ldots, p_N \}$ where $p_i \in \mathbb{R}$ must be chosen appropriately. We seek $P_1$ and $Q_1 > 0$ which satisfy (A.9). Substitution of $P_1$ in (A.9) results in

$$
P_1 (\Delta \otimes K_1) + (\Delta \otimes K_1)^H P_1 = -(P_1 \Delta + \Delta^H P) \otimes \mathcal{I}
$$

Choosing $Q_1 = Q \otimes \mathcal{I}$, the objective is reduced to show that

$$
P_1 \Delta + \Delta^H P = Q
$$

where $Q > 0$. We intend to find $P$ so that $\forall v \in \mathbb{R}^N, v \neq 0, v^T Q v > 0$. Since $\Delta$ is in the Jordan form, $v^T Q v$ can be expressed as

$$
v^T Q v = \sum_{i=1}^{N-1} \text{Re} \{ \lambda_i \} p_i v_i^2 + 2 \sum_{i=1}^{N-2} \rho_i p_i v_i v_{i+1} \quad \text{(D.1)}
$$

in which $\rho_i \in \{ 0, 1 \}$ and $\rho_i = 1$ if $\lambda_i$ is a repeated eigenvalue of $L$. Let $\rho_i = 1$; then one may write:

$$
v^T Q v = \frac{1}{3} \sum_{i=1}^{N-1} \text{Re} \{ \lambda_i \} p_i v_i^2 \\
+ \frac{1}{3} \text{Re} \{ \lambda_1 \} p_1 v_1^2 + \frac{1}{3} \text{Re} \{ \lambda_{N-1} \} p_{N-1} v_{N-1}^2 \\
+ \sum_{i=1}^{N-2} \left( \sqrt{\frac{1}{3} \text{Re} \{ \lambda_{i+1} \}} p_{i+1} v_{i+1} + \frac{p_i}{\sqrt{\frac{1}{3} \text{Re} \{ \lambda_{i+1} \}} p_{i+1}} v_i \right)^2 \\
+ \sum_{i=1}^{N-2} \left( \frac{1}{3} \text{Re} \{ \lambda_i \} p_i - \frac{p_i^2}{\frac{1}{3} \text{Re} \{ \lambda_{i+1} \}} p_{i+1} \right) v_i^2 \quad \text{(D.2)}
$$
Obviously, if \( \rho_i = 0 \), any positive \( p_i \) satisfies (A.9) for \( Q > 0 \). Equation (D.2) will be positive if we set \( \rho_{N-1} = 1 \) and define \( p_i \)'s recursively according to

\[
p_i = \frac{\beta^2}{9} p_{i+1}
\]

for \( i \in \{1, \cdots, N-2\} \). Clearly, the first three lines of (D.2) are positive for \( p_i > 0 \), \( i \in \{1, \cdots, N\} \). We show that the last line is positive for this particular choice. In view of (D.3), we have

\[
\frac{1}{3} \text{Re}\{\lambda_i\}p_i - \frac{p_i^2}{3} \text{Re}\{\lambda_{i+1}\}p_{i+1} = \frac{1}{3} \frac{\beta^2}{9} p_{i+1} \left( \text{Re}\{\lambda_i\} - \frac{\beta^2}{\text{Re}\{\lambda_{i+1}\}} \right) > 0
\]

because \( \text{Re}\{\lambda_i\} > \beta \). Thus, \( P > 0 \) is bounded and

\[
\|P\| = \max\{1, \left(\frac{\beta^2}{9}\right)^{N-1}\} \Rightarrow \|P\| = \|P\|\|K_i^{-1}\|
\]

Moreover, the proposed construction turns out that \( \|Q\| \) is bounded from below since

\[
v^TQv > \frac{1}{3} \sum_{i=1}^{N-1} \beta_i v_i^2 \Rightarrow Q > \frac{1}{3} \beta P
\]

It means that \( \|Q\| = \|Q_1\| > 4\|q\| \) where \( q = O(\beta) \).

### F Simulation Data: Shaping Procedure

#### F.1 Squaring-Down Pre-compensators

There is no need for squaring down agents 1 and 2; to keep the coherency we show

\[
\Sigma_{1,1}: \tilde{u}_1 = u_{p_{1,1}}, \quad \Sigma_{1,2}: \tilde{u}_2 = u_{p_{1,2}}
\]

and \( \Sigma_{1,3} \) and \( \Sigma_{1,4} \) are designed for agents 3 and 4:

\[
\begin{align*}
\Sigma_{1,3}: & \quad \tilde{p}_{1,3} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} p_{1,3} + \begin{bmatrix} 172.1 \\ -165.1 \end{bmatrix} u_{p_{1,3}} \\
\tilde{u}_3 & = \begin{bmatrix} 0.1964 & 0.2411 \\ 1 & 1 \end{bmatrix} p_{1,3} + \begin{bmatrix} -1 \\ 16 \end{bmatrix} u_{p_{1,3}}
\end{align*}
\]

\[
\begin{align*}
\Sigma_{1,4}: & \quad \tilde{p}_{1,4} = -p_{1,4} + 5u_{p_{1,4}} \\
\tilde{u}_4 & = \begin{bmatrix} -1 \\ 1 \end{bmatrix} p_{1,4} + \begin{bmatrix} -1 \\ -15 \end{bmatrix} u_{p_{1,4}}
\end{align*}
\]

#### F.2 Rank-Equalizing Pre-compensators

Agent 1 does not need rank equalization. Thus,

\[
\Sigma_{2,1}: u_{p_{1,1}} = u_{p_{2,1}}
\]

Since agent 2 and 3 are of relative degree 1, the compensators are the same, obviously with different inputs and outputs; so, we just show \( \Sigma_{2,2} \) for agent 2:

\[
\begin{align*}
\Sigma_{2,2}: & \quad \tilde{p}_{2,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} p_{2,2} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{p_{2,2}} \\
\tilde{u}_{p_{1,2}} & = \begin{bmatrix} 1 \\ 0 \end{bmatrix} p_{2,2}
\end{align*}
\]
The compensator for agent 4 is a single integrator

\[ \Sigma_{2,4} : \begin{cases} \dot{p}_{2,4} = u_{p_{2,4}} \\ u_{p_{2,4}} = p_{2,4} \end{cases} \]

F.3 Pre-feedback

Pre-feedback laws can be developed by designing observers for each compensated agent and the following information according to (B.7).

\[
\begin{align*}
R_{d,1} &= 0 \\
R_{a,1} &= 0 \\
R_{d,2} &= \begin{bmatrix} -4 & 2 & 2 \end{bmatrix} \\
R_{a,2} &= 4 \\
R_{d,3} &= \begin{bmatrix} 80 & -24 & 6 \end{bmatrix} \\
R_{a,3} &= \begin{bmatrix} -1 & 0.70 & 0.78 & -1.45 & -4.36 \end{bmatrix} \\
E_{d,4} &= \begin{bmatrix} 72 & -25 & 7 \end{bmatrix} \\
E_{a,4} &= \begin{bmatrix} -1.00 & 1.50 & 1.50 & -4.50 \end{bmatrix}
\end{align*}
\]