Nonlinear Observer for Inertial Navigation Aided by Pseudo-Range and Range-Rate Measurements

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***Abstract*— Range measurement systems are commonly based on measuring time-of-flight of signals encoded in electro-magnetic or acoustic waves. This leads to so-called pseudo-range measurements due systematic errors such receiver clock synchronization error and uncertain wave propagation speed. Inertial navigation aided by pseudo-range measurements is addressed. A modular nonlinear observer is designed and analyzed. The attitude observer is based on a recent nonlinear complementary filter, and includes a gyro bias estimation. The translational motion observer includes the estimation of position, range bias errors (such as receiver clock bias), velocity and specific force, where the latter is used as a reference vector by the attitude observer. The exponential stability of the feedback interconnection of the two observers is analyzed and found to have a semiglobal region of attraction with respect to attitude observer initialization, and local region of attraction with respect to translational motion observer initialization. The latter is due to linearization of the pseudo-range and range-rate measurement equations that is underlying the selection of injection gains using a time varying Riccati equation. In typical applications the pseudo-range and range-rate equations admit an explicit algebraic solution that can be easily computed and used to accurately initialize the position and velocity estimates. Hence, the limited region of attraction is not seen as a limitation of the approach. Advantages of the proposed nonlinear observer include low computational complexity and a solid theoretical foundation.**

I. Introduction

The emergence of low-cost navigation systems is boosted by the availability of low-cost MEMS IMUs (inertial measurement units that provide measurements of acceleration and angular rates), GNSS (global navigation satellite systems), magnetometers, optical sensors, and local range (pseudo-range) sensors based on electro-magnetic and acoustics wave propagation. Our objective is to develop a low-complexity nonlinear observer for inertial navigation aided by a magnetometer and measurements of pseudo-range and range-rate, where the observer has properties founded on stability theory. In general, tightly integrated inertial navigation allows better performance to be achieved than loose integration, in particular in situations with few range measurement, weak or noisy signals, uncertain wave propagation velocity, poor transponder geometry, or other anomalies, e.g. [1], [2].

Commonly used methods for real-time integration or fusion of the data from the individual sensors into position and velocity are nonlinear versions of the Kalman-filter (KF), [1], including the extended KF (EKF), unscented KF, particle filter, and specially tailored variants such as the multiplicative EKF for attitude estimation using quaternions, [3]. While the KF is a general method that has found wide applicability, it does not come without some drawbacks. This includes relatively high computational cost and a rather implicit and not so easily verifiable convergence properties, [4], that may require additional supervisory functions to avoid instability. Its major advantages are flexibility in tuning and application, as it is a widely known and used technology with intuitive and physically motivated tuning parameters interpreted as noise covariances, providing certain optimality properties.

Nonlinear observers for loosely integrated INS aiding have received considerable attention during recent years, and found to have significantly stronger stability properties than nonlinear KF, see e.g. [5], [6], [7], [8]. Still, the use of nonlinear observers has not been much investigated for tight integration. One reason for this is the nonlinear nature of the problem, since the measurement equations are strongly nonlinear in this case, [1], [2]. As a notable exception, tight integration of INS using range or pseudo-range measurements with nonlinear observers is pursued in the series of articles represented by [9], [10], [11], [12]. Using a state transformation and a state augmentation a linear time-varying (LTV) model is derived. This model is closely related to the nonlinear model, and is used for the design of an observer for attitude, position and velocity using hydro-acoustic range measurements. In contrast, our objective is to avoid unnecessary computational complexity.

The proposed nonlinear observer structure is inspired by [7], where a loose integration between GNSS position/velocity measurements and inertial measurements was derived with semiglobal exponential stability conditions. The design philosophy considers that the ranges between the vehicle and the used transponders[1] are relatively large and slowly time-varying. This is a good assumption in many practical situations, such as terrestrial navigation using satellites, and positioning of ships and underwater vehicles using hydro-acoustic transponders at the seabed. In this case, time-varying observer gains multiplying pseudo-range and range-rate errors in the injection terms can be designed to shape the dynamics of the observer using a time-varying linearized relationship between a change in range and change in the vehicle position. Relying on the semi-globally exponentially stable nonlinear attitude observer of [13], [14], we do not use a KF in the observer, and only use the time-varying Riccati

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[1] Note that we use the term transponder as a general concept that also includes radio beacons and navigation satellites in space, for example.

equation for gain matrix updates to the translational motion observer. This allows the integration of the Riccati equation to be performed on a slow time-scale corresponding to the relative motion of the transponders and the receiver, which is typically slow. This leads to low computational complexity. A rigorous analysis of the observer error dynamics stability is made in the paper, where a feature of the proposed approach is a separation into four time scales:

- Instantaneous resetting of position and velocity estimates using an algebraic solution to the pseudo-range equations, [15], [16], during initialization or change of transponder configuration. This approach justifies that only a local region of attraction is required for the position and velocity estimates due to the good initialization accuracy.
- Attitude estimation using a fixed-gain nonlinear observer, including gyro bias, on a fast time-scale driven by the sampling rate of the IMU and magnetometer.
- Estimation of position, velocity, acceleration and bias error model parameters for the pseudo-range measurement system, using a nonlinear observer with time-varying gains operating on a slower time-scale driven by the sampling rate of the pseudo-range measurement system.
- Computation of time-varying gain matrices for the translational motion observer using a Riccati equation. These computations are made on the slowest time-scale driven by the relative motion between the vehicle and the transponders.

A. Notation

We use $||\cdot||_2$ for the Euclidean vector norm, $||\cdot||$ for the induced matrix norm, and denote by $(z_1; z_2)$ the column vector with the vector z_1 stacked over the vector z_2. We denote by I_n the identity matrix of dimension n, and we use 0 to symbolize a matrix or vector of zeros, where the dimensions are implicitly given by the context. For simplicity of notation, we usually let time dependence be implicit.

A quaternion $q = (s_q; r_q)$ consists of a real part $s_q \in \mathbb{R}$ and a vector part $r_q \in \mathbb{R}^3$. For a vector $x \in \mathbb{R}^3$ we denote by $\overline{x}$ the quaternion with zero real part and vector part x, i.e. $\overline{x} = (0; x)$. The Hamilton quaternion product is given by $\otimes$, we let q^* denote the conjugate of the quaternion q, and for a vector $x \in \mathbb{R}^3$ we define the skew-symmetric matrix

$$S(x) \quad := \quad \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

We may use a superscript index to indicate the coordinate system in which a given vector is decomposed, thus x^a and x^b refers to the same vector decomposed in the coordinated systems indexed by a and b, respectively. The rotation between these coordinate systems may be represented by a quaternion q_a^b. The corresponding rotation matrix is denoted $R(q_a^b)$. The rate of rotation of the coordinate system indexed by b with respect to a, decomposed in c, is denoted ω_{ab}^c. We use e for the Earth-Centered Earth-Fixed (ECEF) coordinate system, b for the vehicle BODY-fixed coordinate system, and i for the Earth-Centered Inertial (ECI) coordinate system.

II. Models and preliminaries

A. Vehicle kinematics

The vehicle kinematic model is given by

$$\dot{p}^e = v^e \tag{1}$$

$$\dot{v}^e = -2S(\omega_{ie}^e)v^e + f^e + g^e(p^e) \tag{2}$$

$$\dot{q}_b^e = \frac{1}{2}q_b^e \otimes \overline{\omega}_{ib}^b - \frac{1}{2}\overline{\omega}_{ie}^e \otimes q_b^e \tag{3}$$

where p^e, v^e and f^e are position, velocity and specific force in ECEF, respectively. The attitude of the vehicle is represented by a unit quaternion q_b^e representing the rotation from BODY to ECEF, and ω_{ib}^b represents the rotation rate of BODY with respect to ECI. The known vector ω_{ie}^e represents the Earth's rotation rate about the ECEF z-axis, and $g^e(p^e)$ denotes the plumb-bob gravity vector, which is a function of the vehicle's position.

B. Measurement models

The inertial sensor model is based on the strapdown assumption, i.e. the IMU is fixed to the BODY frame:

$$f_{IMU}^b = f^b \tag{4}$$

$$\omega_{ib,IMU}^b = \omega_{ib}^b + b^b \tag{5}$$

$$\dot{b}^b = 0 \tag{6}$$

where b^b denotes the rate gyro bias that is assumed to satisfy $||b^b||_2 \leq M_b$ for some known bound M_b.

The magnetometer model describes the 3-dimensional Earth magnetic vector field

$$m_{mag}^b = m^b \tag{7}$$

Range measurements are typically generated by measuring the time-of-flight of known signal waveforms (acoustic or electromagnetic) either by some phase-measurement or time of arrival detection circuits. Due to errors in clock synchronization and wave propagation velocity, such measurements often contain systematic errors (biases) in addition to random errors, e.g. [17], and must therefore be treated as pseudo-range measurements. The range measurement model is

$$y_i = \varrho_i + \zeta_i^T \beta, \quad \varrho_i = ||p^e - p_i^e||_2 \tag{8}$$

for $i = 1, 2, ..., m$ where y_i is a pseudo-range measurement, p_i^e is the known position of the i-th transponder, m is the number of measurements, ϱ_i is the geometric range, $\beta \in \mathbb{R}^n$ is a bias error vector to be estimated, and the coefficient vector ζ_i describes the effect of each element of β on range measurement y_i. This framework allows both individual and common mode slowly time-varying errors such as receiver clock bias (i.e. $\zeta_i = 1$ and $\beta := c\Delta_c$ where Δ_c is the clock bias and c is the speed of wave propagation) or uncertain wave propagation speed to be taken into account.

Range-rate measurements are usually found by considering Doppler-shift or tracking of features in signals or sequences of images. Also here there may be systematic (bias) errors in some cases, depending on the sensor principle and technology. The range-rate (speed) measurement model is given by

$$\nu_i = \frac{1}{\varrho_i}\left(p^e - p_i^e\right)^T\left(v^e - v_i^e\right) + \varphi_i^T \beta \tag{9}$$

where ν_i is the relative range-rate measurement, the coefficient vector φ_i describes the effect of each element of β on range speed measurement ν_i, and we define $v_i^e := \dot{p}_i^e$. Eq. (9) follows from time-differentiation of (8), assuming an independent errors model such that we use $\dot{\beta} = 0$ and the term $\varphi_i^T \beta$ instead of $\zeta_i^T \dot{\beta}$ in the model since it provides additional flexibility in modeling. Note that $\dot{\beta} = 0$ is the classical constant parameter assumption in adaptive estimation and does not prevent us from estimating a slowly time-varying β in practice.

We assume that the effect of lever arms, due to sensors being located at different known locations on the vehicle, can be neglected or compensate for by additional rotations and translations.

C. Fundamental properties and assumptions

Despite the nonlinear nature of the pseudo-range measurement equation, we can exploit its quadratic character to get a relatively simple algebraic solution, [15], [16]. Assume an arbitrary position $\hat{p}^e$ is given, and define line-of-sight (LOS) vectors $\breve{p}_i^e := \hat{p}^e - p_i^e$ for every i.

Lemma 1: Assume we have available four pseudo-range measurements y_1, y_2, y_3, y_4 where the three first LOS vectors among $\breve{p}_1^e, \breve{p}_2^e, \breve{p}_3^e, \breve{p}_4^e$ are linearly independent, and

$$y_4 \neq (y_1, y_2, y_3)\breve{A}^{-1}\breve{p}_4^e \tag{10}$$

with

$$\breve{A} = \begin{pmatrix} \breve{p}_1^e & \breve{p}_2^e & \breve{p}_3^e & \breve{p}_4^e \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$$

Assume $\zeta_i = 1$ for all $i = 1, 2, 3, 4$ (i.e. a single common mode error parameter), then $p^e = \hat{p}^e + \tilde{p}^e$ is given by $z = (\tilde{p}^e; \beta)$ where

$$z = \frac{\check{r}\breve{u} + \breve{v}}{2}, \;\; \breve{u} = \breve{A}^{-T}\breve{e}, \;\; \breve{v} = \breve{A}^{-T}\breve{b}$$

$$\check{r} = \frac{-2 - \breve{u}^T W \breve{v} \pm \sqrt{(2 + \breve{u}^T W \breve{v})^2 - \breve{u}^T W \breve{u} \cdot \breve{v}^T W \breve{v}}}{\breve{u}^T W \breve{u}}$$

where $\breve{e} = (1; 1; 1; 1)$, $\breve{b} \in \mathbb{R}^4$ has components $\breve{b}_i = y_i^2 - ||\breve{p}_i^e||_2^2$, and $W = \text{diag}(1, 1, 1, -1)$.

Proof: The proof is similar to [15], [16]. ■

The computations are analytic and the most complex operations are the inversion of a 4×4 matrix as well as the square root computation. We note that there are in general two solutions. This ambiguity can be solved in several ways. One is using five or more range measurements, allowing a linear re-formulation of the estimation problem, [15], [16]. Alternatively, the ambiguity may be resolved using domain knowledge. One example is terrestrial navigation when there is a large distance to the navigation satellites such that non-terrestrial solutions for the vehicle position can be ruled out. Another example is surface or underwater navigation where all transponders are located on the seabed and the vehicle is at the surface or at some distance from the seabed such that positions below the seabed can be ruled out. Altimeters and depth sensors are also useful for this purpose.

Assumption 1: At all time, $\overline{\varrho} \geq \varrho_i \geq \underline{\varrho} > 0$ for all $i = 1, 2, ..., m$.

Assumption 2: At all time, $||v^e - v_i^e||_2 \leq \overline{\nu}$ for all $i = 1, 2, ..., m$.

Assumption 3: The transponder positions p_i^e and velocities v_i^e are known.

Remark 1: We have chosen to consider only the effect of slowly time-varying systematic errors (parameterized by β) in this formulation and analysis. Rapidly varying errors such as high-frequency noise are neglected since they can be handled by appropriate tuning of the gains and may not influence the structure of the observer.

We note that with known position, the range-rate equation (9) becomes linear and thus easy to solve for velocity. In typical range-measurements systems, the remaining measurement errors are typically so small that a good position and velocity initialization of an observer can be found using Lemma 1 such that a relatively small local region of attraction with respect to position and velocity initialization error can be accepted.

III. Nonlinear observer

A. Attitude observer

We use the attitude observer from [13], [14]. It assumes that two vectors have known decompositions in both BODY and ECEF frames, enabling the estimation of the unknown rotation matrix relating the BODY and ECEF frames. More specifically, we will consider the measurements m^b and f^b available in the BODY frame. The corresponding vectors in the ECEF frame is the local magnetic field m^e and some estimate $\hat{f}^e$ that will be an output of the translational motion observer:

$$\dot{\hat{q}}_b^e = \frac{1}{2}\hat{q}_b^e \otimes \left(\overline{\omega}_{ib,IMU}^b - \overline{\hat{b}}^b + \overline{\hat{\sigma}}\right) - \frac{1}{2}\overline{\omega}_{ie}^e \otimes \hat{q}_b^e \tag{11}$$

$$\dot{\hat{b}}^b = \text{Proj}\left(-k_I\hat{\sigma}, \; ||\hat{b}^b||_2 \leq M_{\hat{b}}\right) \tag{12}$$

$$\hat{\sigma} = k_1 m_{mag}^b \times R(\hat{q}_b^e)^T m^e + k_2 f_{IMU}^b \times R(\hat{q}_b^e)^T \text{sat}_{M_f}(\hat{f}^e) \tag{13}$$

where ω_{ie}^e and m^e are assumed known. Observer errors are defined as $\tilde{q} = q_b^e \otimes \hat{q}_b^{e*}$ and $\tilde{b} = b^b - \hat{b}^b$, and we define $\chi = (\tilde{r}; \tilde{b})$ where $\tilde{r}$ denotes the real part of the quaternion $\tilde{q}$. Proj$(\cdot)$ is a projection operator that ensures $||\hat{b}^b||_2 \leq M_{\hat{b}}$ with $M_{\hat{b}} > M_b$, see [7]. Moreover, $\text{sat}_{M_f}(\cdot)$ is a saturation operator, with M_f such that $||f^e||_2 \leq M_f$. The QUEST algorithm, [18], may be used for initialization of the attitude.

Semiglobal stability of the algorithm is established for $\hat{f}^e = f^e$ in [7] under the following assumption:

Assumption 4: The acceleration f^b and its rate $\dot{f}^b$ are uniformly bounded, and there exist a constant $c_{obs} > 0$ such that $||f^b \times m^b||_2 \geq c_{obs}$ for all $t \geq 0$.

B. Translational motion observer

Inspired by [7], we propose the following observer

$$\dot{\hat{p}}^e = \hat{v}^e + \sum_{i=1}^{m} (K_i^{pp} e_{y,i} + K_i^{pv} e_{\nu,i}) \quad (14)$$

$$\dot{\hat{v}}^e = -2S(\omega_{ie}^e)\hat{v}^e + \hat{f}^e + g^e(\hat{p}^e) + \sum_{i=1}^{m} (K_i^{vp} e_{y,i} + K_i^{vv} e_{\nu,i}) \quad (15)$$

$$\dot{\xi} = -R(\hat{q}_b^e)S(\hat{\sigma})f_{IMU}^b + \sum_{i=1}^{m} \left(K_i^{\xi p} e_{y,i} + K_i^{\xi v} e_{\nu,i}\right) \quad (16)$$

$$\hat{f}^e = R(\hat{q}_b^e)f_{IMU}^b + \xi \quad (17)$$

$$\dot{\hat{\beta}} = \sum_{i=1}^{m} \left(K_i^{\beta p} e_{y,i} + K_i^{\beta v} e_{\nu,i}\right) \quad (18)$$

where the gain matrices K_i^* are in general time varying. While the structure is similar to [7], the injection terms are different, and [7] does not include estimation of the parameter vector β. A common feature is that f^e is viewed as an unknown input which is estimated in (16)–(17) to be used in (13). The injection errors from pseudo-range and range-rate measurements are defined as $e_{y,i} := y_i - \hat{y}_i$ and $e_{\nu,i} := \nu_i - \hat{\nu}_i$, with estimated measurements

$$\hat{y}_i = \hat{\varrho}_i + \zeta_i^T \hat{\beta}$$

$$\hat{\nu}_i = \left(\frac{\hat{p}^e - p_i^e}{\hat{\varrho}_i}\right)^T (\hat{v}^e - v_i^e) + \varphi_i^T \hat{\beta}$$

where $\hat{\varrho}_i := ||\hat{p}^e - p_i^e||_2$, and the estimation errors are $\tilde{p} := p^e - \hat{p}^e$, $\tilde{v} := v^e - \hat{v}^e$, and $\tilde{\beta} := \beta - \hat{\beta}$. Next, we consider a linearization of the injection terms, and present the following result that can be proved using Taylor's theorem:

Lemma 2: The injection errors satisfy

$$e_{y,i} = \left(\frac{\hat{p}^e - p_i^e}{\hat{\varrho}_i}\right)^T \tilde{p} + \zeta_i^T \tilde{\beta} + \varepsilon_{y,i} \quad (19)$$

$$e_{\nu,i} = \left(\frac{\hat{v}^e - v_i^e}{\hat{\varrho}_i}\right)^T \tilde{p} + \left(\frac{\hat{p}^e - p_i^e}{\hat{\varrho}_i}\right)^T \tilde{v} + \varphi_i^T \tilde{\beta} + \varepsilon_{\nu,i} \quad (20)$$

where

$$||\varepsilon_{y,i}||_2 \leq \frac{1}{\underline{\varrho}}||\tilde{p}||_2^2 \quad (21)$$

$$||\varepsilon_{\nu,i}||_2 \leq \frac{1}{\underline{\varrho}}||\tilde{p}||_2 \cdot ||\tilde{v}||_2 + \frac{3\overline{\nu}}{2\underline{\varrho}^2}||\tilde{p}||_2^2 \quad (22)$$

We define the state of the error dynamics as $x := (\tilde{p}; \tilde{v}; \tilde{f}; \tilde{\beta})$, where $\tilde{f} := f^e - \hat{f}^e$ replaces ξ as a state by combining (16) and (17). Summarized, the equations for the predicted measurement error can now be written in the linearized time-varying form

$$e_{y,i} = C_{y,i}x + \varepsilon_{y,i} \quad (23)$$

$$e_{\nu,i} = C_{\nu,i}x + \varepsilon_{\nu,i} \quad (24)$$

where the $2m$ rows of the time-varying matrix

$$C := (C_{y,1}; \ldots; C_{y,m}; C_{\nu,1}; \ldots; C_{\nu,m})$$

are defined by $C_{y,i} := (\breve{d}_i^T,\ 0,\ 0,\ \zeta_i^T)$ and $C_{\nu,i} := (\breve{v}_i^T,\ \breve{d}_i^T,\ 0,\ \varphi_i^T)$. The estimated LOS vectors are $\breve{d}_i := (\hat{p}^e - p_i^e)/\hat{\varrho}_i = \breve{p}_i^e/\hat{\varrho}_i$ and the normalized estimated relative velocity vectors are $\breve{v}_i := (\hat{v}^e - v_i^e)/\hat{\varrho}_i$.

We note that the time varying matrix C is known at the current time and can be used for selection of gain parameters. With large distance between the vehicle and transponders, the LOS vector will be slowly time varying, and the velocity will be relatively small. Hence, the measurement matrix C will be slowly time-varying, since due to Lemma 1 the transients resulting from errors in initialization of position and velocity are not expected to be significant.

Following [7] we arrive at the error dynamics

$$\dot{x} = (A - KC)x + \rho_1(t, x) + \rho_2(t, \chi) + \rho_3(t, x) \quad (25)$$

where

$$A := \begin{pmatrix} 0 & I_3 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K := \begin{pmatrix} K_1^{pp} & \ldots & K_m^{pp} & K_1^{pv} & \ldots & K_m^{pv} \\ K_1^{vp} & \ldots & K_m^{vp} & K_1^{vv} & \ldots & K_m^{vv} \\ K_1^{\xi p} & \ldots & K_m^{\xi p} & K_1^{\xi v} & \ldots & K_m^{\xi v} \\ K_1^{\beta p} & \ldots & K_m^{\beta p} & K_1^{\beta v} & \ldots & K_m^{\beta v} \end{pmatrix}$$

The perturbation terms are defined as $\rho_1(t, x) := (0; \rho_{12}(t, x); 0; 0)$ with $\rho_{12}(t, x) := -2S(\omega_{ie}^e)x_2 + (g^e(p^e) - g^e(p^e - x_1))$, and $\rho_2(t, \chi) := (0; 0; \tilde{d}; 0)$ with

$$\tilde{d} = (I - R(\tilde{q})^T)R(q_b^e)(S(\omega_{ib}^b)f^b + \dot{f}^b) - S(\omega_{ie}^e)(I - R(\tilde{q})^T)R(q_b^e)f^b - R(\tilde{q})^T R(q_b^e)S(\tilde{b})f^b$$

see [7], where it is also shown that $||\rho_2(t, \chi)||_2 \leq \gamma_3||\chi||_2$ for some constant $\gamma_3 > 0$. A main difference compared to [7] is that with the range and range-rate measurement model, the matrix C is slowly time-varying, and there is an additional perturbation term $\rho_3(t, x) := K\varepsilon(t, x)$ that results from the linearization of the injection terms: $\varepsilon := (\varepsilon_{y,1}; ...; \varepsilon_{y,m}; \varepsilon_{\nu,1}; ...; \varepsilon_{\nu,m})$. We note that from Lemmas 1 and 2 that the errors are small when $\underline{\varrho}$ is large compared to $||\tilde{p}||_2$, $||\tilde{v}||_2$ and $\overline{\nu}$, and $||\beta||_2$ is relatively small.

As in [7], we want to employ a parameter $\theta \geq 1$ in order to assign a certain time-scale structure to the error dynamics (25). For this purpose, we introduce the non-singular state-transform matrix

$$L_\theta := \text{blockdiag}\left(I_3, \frac{1}{\theta}I_3, \frac{1}{\theta^2}I_3, \frac{1}{\theta^3}I_n\right) \quad (26)$$

and the state transform $\eta = L_\theta x$.

Lemma 3: Let $K_0 \in \mathbb{R}^{(9+n)\times 2m}$ be an arbitrary time-varying gain matrix, and $\theta \geq 1$ be an arbitrary constant. Define

$$K := \theta L_\theta^{-1} K_0 E_\theta \quad (27)$$

where it is assumed that the time-varying $E_\theta \in \mathbb{R}^{2m\times 2m}$ satisfies $E_\theta C = CL_\theta$. Then the error dynamics (25) is equivalent to

$$\frac{1}{\theta}\dot{\eta} = (A - K_0 C)\eta + \frac{1}{\theta}\rho_1(t,\eta) + \frac{1}{\theta^3}\rho_2(t,\chi) + K_0 E_\theta \varepsilon(t, L_\theta^{-1}\eta) \quad (28)$$

Proof: The transformed dynamics are derived by substituting (25) in $\dot{\eta} = L_\theta \dot{x}$. It is straightforward to show that $L_\theta A x = \theta A \eta$. Moreover, $L_\theta K C x = \theta L_\theta L_\theta^{-1} K_0 E_\theta C x = \theta K_0 C L_\theta x = \theta K_0 C \eta$. The rest of the proof follows by change of variables according to $\eta = L_\theta x$. ■

The existence of an E_θ satisfying $E_\theta C = C L_\theta$ depends on the null-space of C. We note that

$$C = \begin{pmatrix} G^T & 0 & 0 & D_p^T \\ B^T & G^T & 0 & D_v^T \end{pmatrix}$$

where $G = (\breve{p}_1^e, ..., \breve{p}_m^e) \in \mathbb{R}^{3\times m}$, $B = (\breve{v}_1^e, ..., \breve{v}_m^e) \in \mathbb{R}^{3\times m}$, and $D = (D_p; D_v)$ with $D_p = (\zeta_1, ..., \zeta_m)$ and $D_v = (\varphi_1, ..., \varphi_m)$. In order to characterize the null-space of C, let $Z \in \mathbb{R}^{n\times(n-k)}$ have $n-k$ columns that forms an orthonormal basis for the null-space of D^T and $Y \in \mathbb{R}^{n\times k}$ have $k = \text{rank}(D^T)$ columns that forms an orthonormal basis for the range-space of D^T. It follows that $D^T Z = 0$ and $\text{rank}(D^T Y) = k$. Consider a vector $x = (x_1; x_2; x_3; x_4)$, where $x_1, x_2, x_3 \in \mathbb{R}^3$ and $x_4 \in \mathbb{R}^n$. Let $x_4 = Z x_{4Z} + Y x_{4Y}$ where $x_{4Z} \in \mathbb{R}^{n-k}$ and $x_{4Y} \in \mathbb{R}^k$. The vector x belongs to the null-space of C if $Cx = 0$, which is equivalent to

$$M \begin{pmatrix} x_1 \\ x_2 \\ x_{4Y} \end{pmatrix} = 0 \quad (29)$$

where

$$M = \begin{pmatrix} G^T & 0 & D_p^T Y \\ B^T & G^T & D_v^T Y \end{pmatrix}$$

Assumption 5: The pseudo-range and range-rate measurement system satisfies:

1) The number of transponders is $m \geq 3 + \lceil k/2 \rceil$.
2) 3 of the estimated LOS vectors are linearly independent, i.e. $\text{rank}(G) = 3$.
3) 3 of the estimated normalized relative velocity vectors are linearly independent, i.e. $\text{rank}(B) = 3$.

Lemma 4: $E_\theta = C L_\theta C^+$ satisfies $E_\theta C = C L_\theta$, where C^+ is the Moore-Penrose right pseudo-inverse of C.

Proof: From Assumption 5 it follows immediately that $M \in \mathbb{R}^{2m\times(6+k)}$ has rank $6+k$, and $2m \geq 6+k$. From (29) it follows that the null-space of C is characterized by $x_1 = 0, x_2 = 0, x_{4Y} = 0$ while x_3 and x_{4Z} can be arbitrary.

Now, consider a singular value decomposition $C = USV^T$, where the Moore-Penrose pseudo-inverse is given by $C^+ = VS^+U^T$. From the characterization of the null-space of C, we have

$$C^+C = VS^+SV^T = \text{blockdiag}(I_3, I_3, 0_3, J)$$

for some matrix $J \in \mathbb{R}^{n\times n}$, and we get $L_\theta C^+ C = C^+ C L_\theta$ due to both C^+C and L_θ sharing the same block diagonal structure. The result follows from

$$E_\theta C = C L_\theta C^+ C = C C^+ C L_\theta = C L_\theta \quad (30)$$

since $CC^+C = C$. ■

Some discussion of Assumption 5 is useful. The assumption is reasonable and closely related to both the assumptions underlying Lemma 1, as well as observability that will be considered shortly. If there are no range-rate measurements then $M = (G^T, D^T) \in \mathbb{R}^{m\times(3+k)}$ and it can be verified that $m \geq 3+k$ transponders, with 3 independent estimated LOS vectors, are needed for M to have full rank.

C. Stability analysis

As the first step towards the stability analysis, we consider the LTV nominal time-scaled error dynamics

$$\frac{1}{\theta}\dot{\eta} = (A - K_0 C)\eta \quad (31)$$

and analyze its stability and robustness before we consider the effect of the perturbations in (28).

Let $R > 0$ be a $2m \times 2m$ symmetric matrix that can be interpreted as the covariance of the pseudo-range and range-rate measurement noises. The observability Gramian for the system $(A, R^{-1/2}C)$ is

$$\mathcal{W}(t, t+T) = \int_t^{t+T} \Phi^T(\tau) C^T(\tau) R^{-1} C(\tau) \Phi(\tau) d\tau$$

where the transition matrix is $\Phi(\tau) = e^{A\tau}$, and we recall (from e.g. [19]) that the LTV system is said to be uniformly completely observable if there exist constants $\alpha_1, \alpha_2, T > 0$ such that for all $t \geq 0$ we have

$$\alpha_1 I \leq \mathcal{W}(t, t+T) \leq \alpha_2 I \quad (32)$$

Assumption 6: The LTV system $(A, R^{-1/2}C)$ is completely uniformly observable.

Remark 2: We note that these assumptions are related to the well-known need for at least three range measurements from transponders at linearly independent positions in order to uniquely determine the three components of the position and velocity vectors, if there are no other unknowns. Independence is related to the well-known requirements that the geometric configuration of the transponders must avoid singularities such as co-linear location, cf. Lemma 1, and the degradation of precision when this configuration is close to singular. When $n = \dim(\beta) > 0$, more than three independent range measurements will be needed, for example it is well known in the context of GNSS receiver clock-bias estimation that a fourth satellite range measurement is needed. In some cases, Assumption 6 follows directly from Assumption 5 and is redundant.

Lemma 5: Suppose

$$K_0 := PC^T R^{-1} \quad (33)$$

where P satisfies the time-scaled Riccati equation

$$\frac{1}{\theta}\dot{P} = PA + A^T P - PC^T R^{-1} CP + Q \quad (34)$$

for some positive definite symmetric matrices R, and $P(0)$, and positive definite symmetric matrix Q. Then P is uniformly bounded and the origin is a globally exponentially

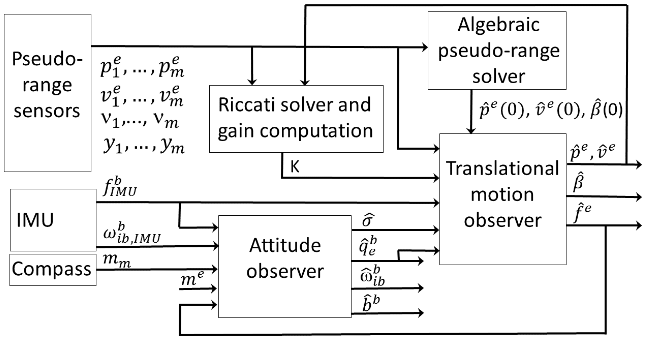

Fig. 1. Observer block diagram.

stable equilibrium point of the LTV nominal error dynamics (31) for any constant $\theta \geq 1$.

Proof: The proof follows from [19], [20], and we repeat the main ideas since we need the Lyapunov function later. Consider a Lyapunov function candidate $U(\eta, t) = \frac{1}{\theta}\eta^T P^{-1}\eta$, which is positive definite and well-defined due to the time-varying matrix P satisfying (34) being symmetric, positive definite with some margin, and bounded, [19]. It follows by standard arguments that along the trajectories of the nominal error dynamics and the solution of (34) that $\dot{U} = -\eta^T(P^{-1}QP^{-1} + C^T R^{-1} C)\eta$. ∎

The structure of the observer is illustrated by the block diagram in Figure 1. We notice two feedback loops where one is due to the use of $\hat{f}^e$ as a reference vector in the attitude observer and the other is caused by linearization of the pseudo-range and range-rate measurement equations to get the C-matrix when solving the Riccati equation for the translational motion observer gain K. Below, we analyze the stability conditions.

Assumption 7: Initial conditions are in the following sets:

- $\mathcal{X} \subset \mathbb{R}^{9+n}$ is a ball containing the origin.
- $\mathcal{P} \subset \mathbb{R}^{(9+n)\times(9+n)}$ is an arbitrary compact set of symmetric positive definite matrices.
- $\mathcal{D}(\bar{\varepsilon}) = \{\tilde{q} | \ |\tilde{s} > \bar{\varepsilon}\}$ represents a set of attitude errors bounded away from 180° by a (small) margin determined by an arbitrary constant $\bar{\varepsilon} \in (0, \frac{1}{2})$.
- $\mathcal{B} = \{b \in \mathbb{R}^3 \mid ||b||_2 \leq M_b\}$.

Assumption 8: Observer gains are chosen according to

- $k_1, k_2 > 0$ are sufficiently large, cf. [7].
- $k_I > 0$ is arbitrary.
- K is chosen according to (27), (33) and (34) tuned by symmetric $R > 0$ and symmetric $Q > 0$.

Proposition 1: There exists a $\theta^* \geq 1$ such that for all $\theta \geq \theta^*$ there exist a non-empty ball $\mathcal{X}$ such that P is uniformly bounded and

$$\sqrt{||x(t)||_2^2 + ||\chi(t)||_2^2} \leq \kappa e^{-\lambda t}\sqrt{||x(0)||_2^2 + ||\chi(0)||_2^2}$$

for some $\kappa > 0$ and $\lambda > 0$.

Proof: Using $U(\eta, t) := \frac{1}{\theta}\eta^T P^{-1}\eta$, we get from the proof of Lemma 5 that

$$\begin{aligned} \dot{U} &= -\eta^T(P^{-1}QP^{-1} + C^T R^{-1} C)\eta \\ &\quad + \frac{2}{\theta}\eta^T P^{-1} P C^T R^{-1} E_\theta \varepsilon \\ &\quad + \frac{2}{\theta}\eta^T P^{-1}\rho_1(t,\eta) + \frac{2}{\theta^3}\eta^T P^{-1}\rho_2(t,\chi) \\ &\leq -\gamma_1 ||\eta||_2^2 \\ &\quad + \frac{2}{\theta}||\eta||_2 \cdot ||C^T R^{-1}|| \cdot \sum_{i=1}^{m} ||E_\theta||(\varepsilon_{y,i}^2 + \varepsilon_{r,i}^2) \\ &\quad + \frac{1}{\theta}\gamma_2\gamma_4||\eta||_2^2 + \frac{1}{\theta^3}\gamma_3\gamma_4||\eta||_2 \cdot ||\chi||_2 \end{aligned} \tag{35}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$ are constants independent of θ. Note that a uniform upper bound on P^{-1} that does not depend on θ exists, see Lemma 6. Using Lemma 2,

$$\begin{aligned} \dot{U} &\leq -\gamma_1||\eta||_2^2 + \frac{1}{\theta}\gamma_5(\underline{\varrho}, \overline{\nu})||\eta||_2^3 \\ &\quad + \frac{1}{\theta}\gamma_2\gamma_4||\eta||_2^2 + \frac{1}{\theta^3}\gamma_3\gamma_4||\eta||_2 \cdot ||\chi||_2 \end{aligned}$$

where $\gamma_5(\underline{\varrho}, \overline{\nu})$ increases with $\overline{\nu}$ and decreases with $\underline{\varrho}$, and is independent of θ.

Similar to [7], we can show that for any $\delta > 0$ and $T > 0$ there exists a $\theta_1^* \geq 1$ such that for $\theta \geq \theta_1^*$ there exists an invariant set $\mathcal{X}_1 \subset \mathbb{R}^{9+n}$ such that for $||\eta(0)||_2 \in \mathcal{X}_1$ we have for all $t \geq T$ that $||\eta||_2 \leq \delta$. As argued in [7] this implies $|\tilde{s}| \geq \bar{\varepsilon}$ such that $\tilde{q}$ never leaves $\mathcal{D}(\bar{\varepsilon})$. Inspired by [7], we now define the function

$$W(t, \tilde{r}, \tilde{s}, \tilde{b}) \;:=\; \left(1 - \tilde{s}^2\right) + 2\ell\overline{\tilde{s}\tilde{r}}R(q_b^e)\tilde{b} + \frac{\ell}{k_I}\tilde{b}^T\tilde{b}$$

where $\ell > 0$ is a constant [14]. Under the condition $|\tilde{s}| \geq \bar{\varepsilon}$, W is shown in [7] to satisfy

$$\dot{W} \;\leq\; -\gamma_7||\chi||_2^2 + \gamma_6\theta^2||\chi||_2 \cdot ||\eta||_2 \tag{36}$$

for some constants $\gamma_6, \gamma_7 > 0$ that are independent of θ. Next we define the Lyapunov-function candidate $V(t, \eta, \chi) := U(t, \eta) + \frac{1}{\theta^5}W(t, \chi)$. Then

$$\dot{V} \leq -z\begin{pmatrix} \gamma_1 - \frac{\gamma_2\gamma_4}{\theta} & -\frac{\gamma_3\gamma_4 + \gamma_6}{2\theta^3} \\ -\frac{\gamma_3\gamma_4 + \gamma_6}{2\theta^3} & \frac{\gamma_7}{\theta^5} \end{pmatrix} z + \frac{\gamma_5(\underline{\varrho}, \overline{\nu})}{\theta}||\eta||_2^3$$

where we have defined the auxiliary state $z := (||\eta||_2; ||\chi||_2) \in \mathbb{R}^2$. Considering the first- and second-order principal minors, we get the conditions

$$\theta^* > \max\left(\frac{\gamma_2\gamma_4}{\gamma_1}, \frac{\gamma_2\gamma_4\gamma_7 + (\gamma_3\gamma_4 + \gamma_6)^2}{\gamma_1\gamma_7}\right) \tag{37}$$

Hence, we can choose a $\theta^* \geq \theta_1^*$ satisfying (37) such that for all $\theta \geq \theta^*$ there exists an invariant set $\mathcal{X}_2$ and $\alpha_3, \alpha_4 > 0$, where for all $x \in \mathcal{X}_2$ we have

$$\dot{V} \leq -\alpha_3||z||_2^2 - \alpha_4||\chi||_2^2 \leq -2\lambda V$$

for some $\lambda > 0$, and the result follows by choosing $\mathcal{X}$ as the largest invariant set such that $\mathcal{X} \subset \mathcal{X}_1 \cap \mathcal{X}_2$, and application of the comparison lemma, [21]. ∎

In typical applications where the LTV system (28) is slowly time varying, we can reap the benefits of solving the

Riccati equation on a slower time-scale than the estimator updates, roughly speaking only when there is a significant change in the C-matrix due to the relative motion of the vehicle and the transponders, or enabling or disabling some range measurements.

IV. Case study

We consider tight integration of a GNSS using pseudo-range measurements, an IMU and a magnetometer. We consider four satellites with initial positions based on an example in [1], p. 271. The initial receiver clock bias leads to an initial range error $\beta(0) = c\Delta_c = 264691.13\ m$, and we assume the clock drifts with an effect $\dot{\beta} = 0.001\ m/s$. We simulate a smooth motion with some relatively small accelerations and angular velocities.

We simulate the pseudo-range measurements with $1\ s$ sampling period with white noise of $1\ m$ standard deviation. Magnetometer, gyro and accelerometers are sampled at $10\ ms$ with simulated random noise with standard deviations of $1.15\ mGauss$, $0.8\ deg/s$ and $0.09\ m/s^2$, respectively, corresponding to the typical RMS output noise specification of an Analog Devices ADIS 16407 IMU. The gyro has a constant unknown bias. Observer gains are chosen as $k_I = 0.03$, $k_1 = k_2 = 0.5$, $\theta = 1$, $R = I_4$, and $Q = \mathrm{diag}(0, 0, 0, 10^{-7}, 10^{-7}, 10^{-7}, 0.25, 0.25, 0.25, 5) \cdot 10^{-3}$. The Riccati equation is initialized by $P(0)$ solving the algebraic Riccati equation. Euler's method is used to discretize the observer and to simulate the vehicle's motion.

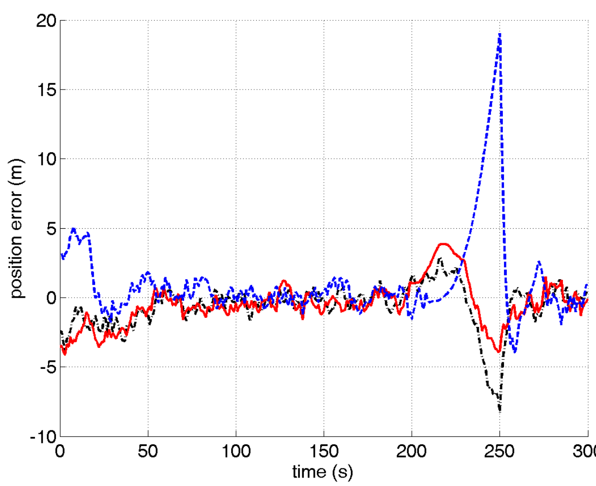

Fig. 2. Simulation results wih errors in position. Black (dashed-dotted) is x-component, red (solid) is y-component, while blue (dashed) is z-component.

Typical simulation results are show in Figures 2–5. Initial values $\hat{p}^e(0)$ and $\hat{\beta}(0)$ are computed using Lemma 1 at time $t = 0$. Initial values for the attitude estimates are set to a few degrees error, and $\hat{v}^e(0) = 0$, $\xi(0) = 0$ and $\hat{b}(0) = 0$.

Between $200\ s \leq t \leq 250\ s$ the signal from Satellite 3 is unavailable, and between $210\ s \leq t \leq 230\ s$ the signal from Satellite 4 is also unavailable. Hence, the observability condition is violated during $200\ s \leq t \leq 250\ s$. We observe

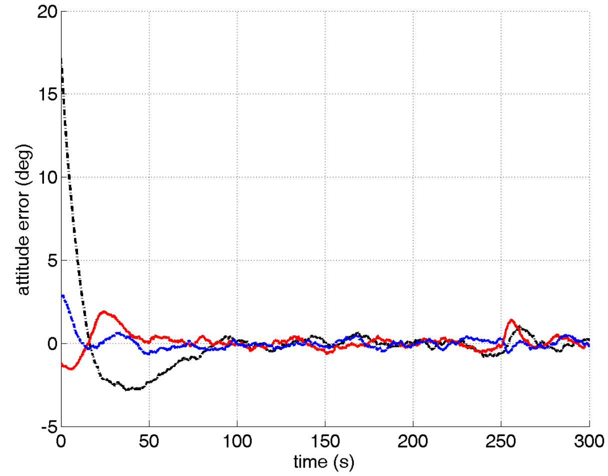

Fig. 3. Simulation results. Errors in attitude. Black (dashed-dotted) is yaw, red (solid) is pitch, while blue (dashed) is roll.

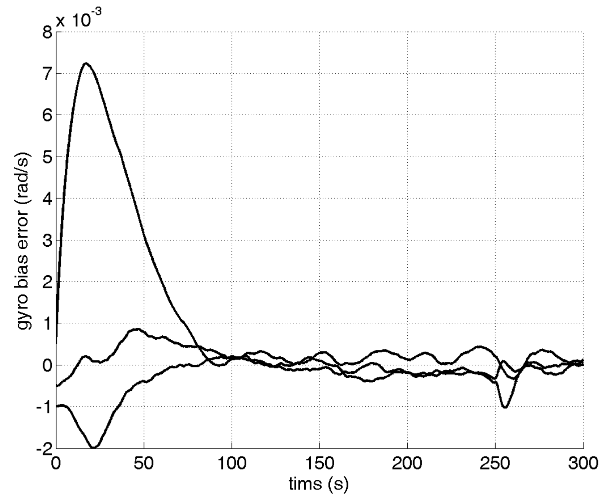

Fig. 4. Simulation results. Error in gyro bias estimates.

Fig. 5. Simulation results. Error in clock bias estimate.

that mainly the z-component of the position and velocity estimates are influenced and drifts off. We note however that degradation of performance is quite graceful and that the observer recovers quickly without any resetting.

During the 300 s simulation the velocity of the satellites leads to changes in the C matrix that lead to changes in the gain matrix K, typically up to 20 %. It is verified that updating P and the gain K by solving the algebraic Riccati equation only every 30 s does not lead to any noticeable loss of performance.

The computational complexity of the attitude observer is a total of 138 arithmetic operations (additions and multiplications). This is the main contribution to computational complexity, as these computations are scheduled at 100 Hz. In comparison, the translational motion observer is updated at 1 Hz, which requires 249 arithmetic operations. It should be mentioned that the Riccati-equation update has a computational complexity of the same order of magnitude, but is sufficient to update only at a rate of approximately $0.03Hz$ or less.

V. Conclusions

Position estimation based on range- and pseudo-range measurements is an inherently nonlinear problem with non-unique solutions due to the spherical topology involved. In order to design an estimator for fusing the range measurements with inertial and compass measurements, we have designed a nonlinear observer where the only linearization is made with respect to the pseudo-range measurement equations. The practical validity of the linearization is strongly motivated by the fact that a computationally simple and explicit formula can be used to explicitly solve the pseudo-range equations in order to accurately initialize the nonlinear observer position and velocity estimates. The observer has exponential stability in a region of attraction that is semi-global with respect to attitude initialization, and local with respect to initialization of position and velocity. In contrast, an EKF may have local region of attraction both with respect to initialization of all states. A key feature of the method is a time-scale separation that allows different observer blocks to be updated at different rates, cf. Figure 1. This means that the total nonlinear observer is computationally efficient.

Acknowledgments

Thanks to Torleiv Bryne for comments and discussions. This work was supported by the Research Council of Norway, Statoil, DNV GL and Sintef through the Centers of Excellence funding scheme, Grant 223254 - Center for Autonomous Marine Operations and Systems (AMOS) and the Research Council of Norway through grant 221666.

Appendix A

Lemma 6: Consider P, the solution of (34). There exists a uniform upper bound on P^{-1} that is independent of $\theta \geq 1$.

Proof: We have that (34) is equivalent to

$$\frac{1}{\theta}\dot{P}^{-1} + P^{-1}A + A^T P^{-1} - C^T R^{-1} C + P^{-1} Q P^{-1} = 0$$

Let $X = P^{-1}$ with $X(0) = P^{-1}(0)$:

$$\frac{1}{\theta}\dot{X} = -XA^T - A^T X + C^T R^{-1} C - XQX \quad (38)$$

It is well known that $X(t) > 0$ for all t. For $||X||$ large, the last term in (38) will dominate the other terms. Hence, there exists an $\alpha > 0$ such that $\frac{d}{dt}||X|| < 0$ for all $||X|| > \alpha$. By choosing α sufficiently large, $X^{-1}(0) = P(0) \in \mathcal{P}$ also belongs to this invariant set. The proof is complete by observing that the bound α does not depend on θ. ■

References

[1] J. A. Farrell, *Aided Navigation. GPS with high rate sensors*. McGraw-Hill, 2008.

[2] P. D. Groves, *Principles of GNSS, Inertial, and Multisensor Integrated Navigation Systems*, 2nd ed.

[3] F. Markley, "Attitude error representation for kalman filtering," *J. Guidance, Control and Dynamics*, vol. 26, pp. 311–317, 2003.

[4] K. Reif, F. Sonnemann, and R. Unbehauen, "An EKF-based nonlinear observer with a prescribed degree of stability," *Automatica*, vol. 34, pp. 1119–1123, 1998.

[5] B. Vik and T. I. Fossen, "A nonlinear observer for GPS and INS integration," in *Proc. IEEE Conf. Decision and Control, Orlando, FL*, 2001, pp. 2956–2961.

[6] M.-D. Hua, "Attitude estimation for accelerated vehicles using GPS/INS measurements," *Control Engineering Practice*, vol. 18, pp. 723–732, 2010.

[7] H. F. Grip, T. I. Fossen, T. A. Johansen, and A. Saberi, "Nonlinear observer for GNSS-aided inertial navigation with quaternion-based attitude estimation," in *Proc. American Control Conference, Washington DC*, 2013.

[8] ——, "Globally exponentially stable attitude and gyro bias estimation with application to GNSS/INS integration," *Automatica*, vol. 51, pp. 158–166, 2015.

[9] M. Morgado, P. Batista, P. Oliveira, and C. Silvestre, "Position and velocity USBL/IMU sensor-based navigation filter," in *Proc. 18th IFAC World Congress, Milan, Italy*, 2011, pp. 13 642–13 647.

[10] P. Batista, C. Silvestre, and P. Oliveira, "Sensor-based long baseline navigation: observability analysis and filter design," *Asian J. Control*, vol. 16, pp. 974–994, 2014.

[11] P. Batista, "GES long baseline navigation with unknown sound velocity and discrete-time range measurements," *IEEE Trans. Control Systems Technology*, vol. 23, pp. 219–230, 2015.

[12] P. Batista, C. Silvestre, and P. Oliveira, "Tightly coupled long baseline/ultra-short bseline integrated navigation systems," *Int. J. Systems Science*, 2015, to appear.

[13] R. Mahoney, T. Hamel, and J.-M. Pfimlin, "Nonlinear complementary filters on the special orthogonal group," *IEEE Trans. Automatic Control*, vol. 53, pp. 1203–1218, 2008.

[14] H. F. Grip, T. I. Fossen, T. A. Johansen, and A. Saberi, "Attitude estimation using biased gyro and vector measurements with time-varying reference vectors," *IEEE Trans. Automatic Control*, vol. 57, pp. 1332–1338, 2012.

[15] S. Bancroft, "An algebraic solution to the GPS equations," *IEEE Trans. Aerospace and Electronic Systems*, vol. 21, pp. 56–59, 1985.

[16] J. Chaffee and J. Abel, "On the exact solutions of pseudorange equations," *IEEE Trans. Aerospace and Electronic Systems*, vol. 30, pp. 1021–1030, 1994.

[17] D. Dardari, E. Falletti, and M. Luise, *Satellite and Terrestrial Radio Positioning Techniques*. Academic Press, 2012.

[18] M. Shuster and S. Oh, "Three-axis attitude determination for vector observations," *J. Guidance, Control and Dynamics*, vol. 4, pp. 70–77, 1981.

[19] R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *Trans. ASME, Ser. D, J. Basic Eng*, pp. 95–109, March 1961.

[20] B. D. O. Anderson, "Stability properties of Kalman-Bucy filters," *J. Franklin Institute*, vol. 291, pp. 137–144, 1971.

[21] H. K. Khalil, *Nonlinear systems*. Prentice-Hall, 2002.