Three-stage filter for position estimation using pseudo-range measurements

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***Abstract*—The estimation of a vehicle's position and velocity based on measurements of pseudo-ranges and range-rates is a nonlinear filtering problem. We utilize a linearized Kalman-filter with global convergence that avoids divergence since the local model linearization is not based on the linearized Kalman-filter's own estimates. Instead, the linearization is made about the estimates of another sub-optimal Kalman-filter that is based on a globally valid quasi-linear time-varying measurement model.**

I. Introduction

A. Background and motivation

Range measurement systems for vehicle navigation typically involve the deployment of some kind of sensor on the vehicle that recognizes signals from multiple transponders at known locations. Here we use the term transponder in a generalized way, meaning anything from an acoustic transponder, to a navigation satellite in space, radio beacons, or optical/laser instruments. The sensor then measures signal time of arrival (TOA), phase difference, or some other variable that is a monotonic function of the range. The geometric range so obtained is a nonlinear function of vehicle and transponder positions given by their Euclidean distance. Since the geometric range is often not measured directly, one often needs to employ a pseudo-range measurement model that includes a slowly time-varying bias parameter due to unknown clock synchronization errors, or other error sources. The problem of estimating the vehicle position, in some reference coordinate frame, is therefore a highly nonlinear estimation problem that is further complicated by the need to estimate the bias parameter.

At least four measurements are needed to estimate the four variables: three position coordinates and a bias parameter. With a sufficiently large number of pseudo-range measurements, the global nonlinear algebraic measurement equations of the position estimation problem generally has two solutions. An exception is when there are redundant measurements, in which case, the solution is usually unique, see [1], [2] and the references therein. Such measurement systems are often referred to as TOA range measurements, or time-difference-of-arrival (TDOA) range measurements after the unknown clock bias has been eliminated from the equations by taking the difference between the TOA measurements. The same mathematical problem also arises in other problems and has been extensively studied in source localization where multiple receivers measure the TOA or TDOA from a single source, see [3], [4].

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B. Literature review

Many approaches to the design of estimators for navigation based on pseudo-range measurements are available. Some of these approaches use a **local linearization (first order Taylor series approximation) of the measurement equation** as the basis for a nonlinear least squares, maximum likelihood or approximate nonlinear Kalman-filter (KF), see e.g. [1], [2], [5], [6]. While these methods provide solutions that have been very successful in numerous of applications, they also have drawbacks such as implicit convergence conditions. Their convergence is not generally guaranteed when sufficiently accurate initialization cannot be made, when the vehicle is operating near or across transponder baselines, or in singular transponder configurations. This is less of a problem for vehicles operating near the Earth's surface using satellite navigation, or underwater vehicles using hydro-acoustic navigation with seabed or surface transponders. However, these nonlinearities should be carefully dealt with in other applications where the transponders are deployed locally and the vehicle's position relative to the transponders may be, more or less, arbitrary. Such applications include robotic and vehicular systems for terrestrial, low altitude, and indoor navigation and localization systems.

Despite the highly nonlinear nature of the problem, it is known [1], [2] that **globally valid nonlinear algebraic transformations** of the hyperbolic pseudo-range equations can be used to derive a set of quasi-linear equations. This quasi-linear time-varying measurement model is free of approximations in the noise-free case, and can be used to define low-complexity estimators in closed form using least-squares or similar linear algebraic techniques. Such models are called ‘quasi-linear’ since they have been algebraically manipulated into linear form by adding new variables, or by eliminating certain nonlinear (quadratic) relationships that exist in the data. It is also well known that the explicit algebraic transformation leads to quasi-linear equations whose direct least-squares equation-error solution gives sub-optimal estimates since the prior information about the nonlinearities has been removed. In particular the estimators do not achieve the Cramer-Rao lower bound, [3], [7], [8]. Some methods employ appropriately weighted least-squares methods to recover performance, [9], [10] or directly include the quadratic terms in the estimation problem [11], [7], [12].

An interesting approach is pursued in a series of articles represented by [13], [14], [15]. Using a **state transformation and state augmentation**, they derive a different quasi-linear time-varying (LTV) model whose output is equivalent to the nonlinear model in the noise-free case. This is then used to design a KF for position and velocity using range measurements.

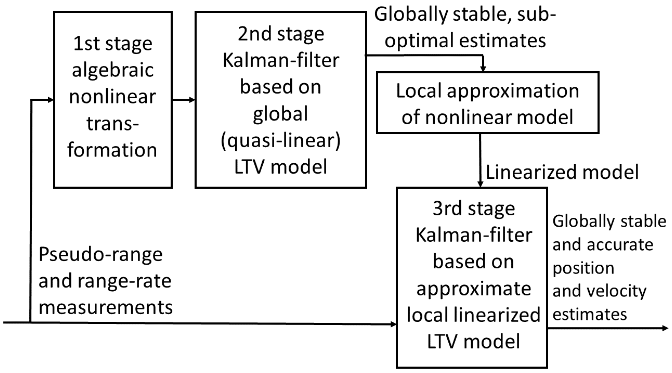
Two-stage approaches using quasi-linear models resulting from such nonlinear algebraic transforms have been proposed previously. Some methods improve the accuracy of the estimates by combining the quasi-linear global solution approach with various re-design techniques that employ the neglected quadratic relationship to improve the estimates in a second stage that uses local linearization, [16], [9], [17], [8]. The two-stage estimator proposed by [18] employs an algebraic solution to the transformed quasi-linear measurement equations in a first stage, and a KF for reducing the noise in a second stage. Since the stochastic properties of the measurements are highly influenced by the nonlinear transform and algebraic solution, the tuning of the KF covariance matrices is not straightforward and the achieved accuracy will suffer from this. Similarly, [19] proposes a loosely coupled integrated INS and GNSS where the GNSS receiver is assumed to solve its position using algebraic computations based on the transformed quasi-linear measurement model, and a Kalman-filter is used for integration of the INS and GNSS data. A similar strategy was taken in [20], where attitude is also estimated and nonlinear observers are used instead of the KF.

C. Main contribution

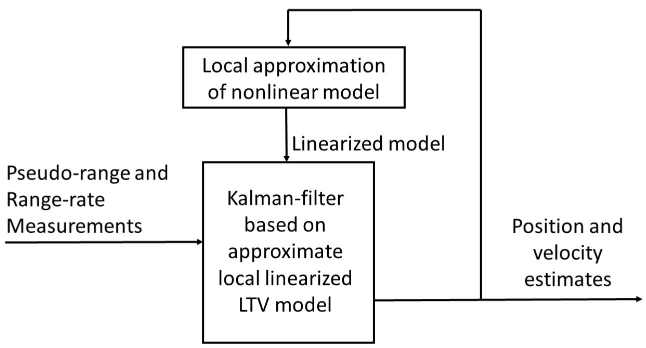
The extension of two-stage approaches to a uniformly globally asymptotically stable (UGAS) **three-stage estimation strategy** with a final stage linearized KF and dynamic vehicle model is the main novelty and contribution of the present work.

Our goal is to reap the benefit of a globally unique and convergent solution resulting from the algebraic transformation without sacrificing estimation accuracy in the context of dynamic vehicle navigation. We first employ a global nonlinear transformation of the algebraic measurement equations and variables into the quasi-linear form. In a 2nd stage, a globally convergent estimate is generated by combining the quasi-linear measurement model and data from the 1st stage with a vehicle dynamics model using a standard KF. Note that unlike [18], [19], [20], the first stage does not generate an algebraic estimate, but merely transforms the measurements such that the 2nd stage estimator can be designed based on a globally valid LTV model. An advantage is that the covariances of the calculated measurements explicitly follows from the variances of the original measurements such that the tuning of the 2nd stage KF is rather straightforward. However, the estimate resulting from this KF is still sub-optimal in the presence of measurement noise, since prior information about the nonlinear relationship has been eliminated by the nonlinear transformation in order to get a global quasi-linear model. We therefore use those 2nd stage estimates to determine a point about which a first order Taylor-series approximation of the original nonlinear measurement model can be computed. The resulting locally linearized time-varying model is then used in a 3rd stage KF to recover more accurate estimates. These are uniformly globally asymptotically stable (UGAS) due to the cascading of the two UGAS KFs and the algebraic nonlinear transformation. The scheme is shown in schematic form in Figure 1, where it can also be compared to the well-known Extended KF (EKF) to see how the potentially de-stabilizing nonlinear feedback loop of the EKF is avoided by a cascade (feed-forward) interconnection. Note that the main distinction between the proposed approach and the EKF is which state estimates are used for linearization in the final stage, which is the only stage in the EKF.

In contrast to [13], [14], [15], the scheme described in the current paper fully exploits the quadratic measurement model and thereby avoid the state augmentation for the benefit of simple tuning and low computational complexity. This is particularly attractive when there are redundant (more than four) independent range measurements. Moreover, the observability conditions we require are simple and intuitive. We can also take advantage of range-rate measurements and address sub-optimality issues that appear in the presence of measurement noise, through the 3rd stage KF.



(a) Three-stage estimator cascade.



(b) Extended Kalman Filter.

Fig. 1. Proposed three-stage estimator, and its relationship to the extended Kalman filter.

D. Organization of the paper

The layout of the remainder of the paper is as follows: In section II, we describe the nonlinear measurement models. In section III we present the first stage pre-processing in terms of algebraic transformation of the hyperbolic pseudo-range and range-rate equations into quasi-linear form. In section IV this quasi-linear model is combined with a model for vehicle dynamics and used to design a uniformly globally asymptotically stable 2nd stage KF for estimation of position, velocity and range measurement bias. The 3rd stage linearized KF for recovery of performance is described in section V. Section VI illustrates the global convergence and performance achieved by the three-stage estimation algorithm. Comparisons with the EKF are also presented before conclusions are presented in Section VII.

E. Notation

$\mathbf{R}$ is the set of real numbers, and we use $||\cdot||_2$ for the Euclidean vector norm, $||\cdot||$ for the induced matrix norm, and denote by $(z_1; z_2)$ the column vector with the vector z_1 stacked over the vector z_2. I_n is the identity matrix of dimension n, and we use 0 to symbolize a matrix of zeros, where the dimensions are implicitly given by the context. For a matrix A, we use A^+ to denote the left weighted Moore-Penrose pseudo-inverse (given by $A^+ = (A^T W A)^{-1} A^T W$ for the case when A has full column-rank and $W = W^T > 0$ is a weighting matrix, see [21] for the general definition using the singular value decomposition). $\mathbf{E}$ denotes statistical expectation, and the operator q^{-1} denotes the one-sample delay operator. For simplicity of notation, we usually let time dependence be implicit.

II. Measurement models

Let $p^e \in \mathbf{R}^3$ and $v^e \in \mathbf{R}^3$ denote position and velocity, respectively. The superscript e refers to an Earth-fixed Cartesian coordinate frame such as East-North-Up (ENU) or Earth-Centered-Earth-Fixed (ECEF). Assuming direct line-of-sight (LOS) path between transponders and receiver, the range measurement model is

$$y_i = \varrho_i + \beta, \quad \varrho_i = ||p^e - p_i^e||_2 \tag{1}$$

for $i = 1, 2, ..., m$ where y_i is a pseudo-range measurement, p_i^e is the known position of the i-th transponder, m is the number of measurements, ϱ_i is the geometric range along the LOS, and $\beta \in \mathbf{R}$ is a common error (bias) to be estimated. This framework incorporates slowly time-varying common mode errors such as the receiver clock synchronization error (in this case $\beta := c\Delta_c$ where Δ_c is the clock synchronization error and c is the speed of signal propagation). For simplicity of notation we let white measurement noise be implicit when presenting the measurement models, algebraic solutions and the nonlinear transforms in the first stage of the filter, and return to consider this noise more explicitly when using the Kalman filters in the second and third stages For discussion of other error sources, such as multi-path and non-line-of-sight, see [1].

The range-rate (speed) measurement model is given by

$$\nu_i = \dot{y}_i = \frac{1}{\varrho_i} (p^e - p_i^e)^T (v^e - v_i^e) + \gamma \tag{2}$$

where ν_i is the relative range-rate measurement. We define $v_i^e := \dot{p}_i^e$. The first term of (2) follows from time-differentiation of (1). The error, γ, depends on the range-rate measurement system. If it is directly derived from the range measurements, y_i, then $\gamma = \dot{\beta}$ may be a suitable model. If the range rates are measured using an independent principle such as Doppler shift, then γ can be viewed as being independent of β.

The following assumptions are made on the measurement system:

Assumption 1. *Transponder positions p_i^e and their velocities v_i^e are known (and may be time-varying).*

Assumption 2. *There are available $m \geq 4$ pseudo-range measurements $y = (y_1; y_2; ...; y_m)$.*

Assumption 3. *The bias parameter β is modelled as a random walk process:*

$$\dot{\beta} = u_\beta \tag{3}$$

where u_β is white noise.

We remark that for stationary transponders, their position can be computed from range measurements in a calibration procedure where the vehicle is at standstill at a known reference position.

III. Global Quasi-Linear Model and First Stage Pre-Processing

Despite the nonlinear nature of the measurement equations (1) and (2), it is known that one can exploit their quadratic character to obtain a relatively simple algebraic solution, or even transform them into a quasi-linear form, see [1], [2] and the references therein. In this section, we review these well known results.

Let a known fixed reference position p_0^e be chosen, and define the transponder LOS vectors $\breve{p}_i^e := p_i^e - p_0^e$ relative to this reference position for every i. From (1) we obtain

$$(y_i - \beta)^2 = (p_\Delta^e - \breve{p}_i^e)^T (p_\Delta^e - \breve{p}_i^e) \tag{4}$$

where we have defined $p_\Delta^e := p^e - p_0^e$. Expanding and rearranging terms yields

$$y_i^2 - ||\breve{p}_i^e||_2^2 = 2\left[-(\breve{p}_i^e)^T \quad y_i\right] x - \beta^2 + ||p_\Delta^e||_2^2 \tag{5}$$

where $x := (p_\Delta^e; \beta)$. Note that stacking instances of (5) for $i = 1, 2, ..., m$ into matrices and vectors gives

$$z = 2C_{zw}x - r\ell \tag{6}$$

where $\ell := (1; ...; 1) \in \mathbf{R}^m$, $z \in \mathbf{R}^m$ has components $z_i := y_i^2 - ||\breve{p}_i^e||_2^2$. We have also introduced an auxiliary variable

$$r := \beta^2 - ||p_\Delta^e||_2^2 = -x^T M x \tag{7}$$

with $M := \mathrm{diag}(1, 1, 1, -1)$, and the matrix $C_{zw} \in \mathbf{R}^{m \times 4}$

$$C_{zw} \quad := \quad \begin{bmatrix} -(\breve{p}_1^e)^T & y_1 \\ \vdots & \\ -(\breve{p}_m^e)^T & y_m \end{bmatrix}$$

The direct algebraic solution to the measurement model (6) and (7) for position is presented next. It will be partly used in the estimation algorithm, and it also provides some insight into the structure of the solution and observability conditions that will be useful later.

Lemma 1. *Suppose rank$(C_{zw}) = 4$. Consider the two candidate solutions $(x; r)$ given by*

$$x = \frac{rc + w}{2}, \; c = C_{zw}^+ \ell, \; w = C_{zw}^+ z \tag{8}$$

$$r = \begin{cases} \frac{-h \pm \sqrt{h^2 - w^T M w \cdot c^T M c}}{c^T M c}, & \text{if } c^T M c \neq 0 \\ -\frac{w^T M w}{2h}, & \text{otherwise} \end{cases} \tag{9}$$

where $h := 2 + w^T M c$. Provided the following condition is satisfied

$$e = (C_{zw} C_{zw}^+ - I_m)(r\ell + z) = 0, \tag{10}$$

then the position solution $p^e = p_0^e + p_\Delta^e$ (where p_Δ^e is extracted from x) solves the pseudo-range equations (1). Moreover, at least one of the two alternative candidate solutions (8)-(9) is a valid solution satisfying (10) and is equal to the true position.

Proof. From (8) we obtain

$$\begin{aligned} 2C_{zw}x &= 2C_{zw}\left(\frac{rC_{zw}^+\ell + C_{zw}^+ z}{2}\right) \\ &= C_{zw}C_{zw}^+(r\ell + z) \\ &= (r\ell + z) + e \end{aligned}$$

We can conclude that the necessary and sufficient condition given by (6) is satisfied since $e = 0$ due to (10). Substituting $x = (rc + w)/2$ into (7) gives

$$-4r \quad = \quad (rc + w)^T M (rc + w) \tag{11}$$

Solving (11) for r gives (9).

Finally, we note that the pseudo-range measurement equations always have at least one solution. We observe that $C_{zw}^T C_{zw} \in \mathbf{R}^{4\times4}$ is non-singular by the rank assumption. Hence, application of the pseudo-inverse results in a unique solution that must be a valid solution for x if r satisfies (10). Hence, there exists at least one valid solution. □

For the case $m = 4$ we always have $e = 0$ due to $C_{zw}^+ C_{zw} = I_4$ since $C_{zw}^T C_{zw}$ is non-singular. Hence, we have two solutions both satisfying (10), except in the degenerate case $c^T M c = 0$ where there is a single solution. For $m \geq 5$ we generally have a single solution satisfying (10), except in degenerate cases where there may be two solutions. However, instead of using (8), (9), and (10), this unique solution can be found in a simpler, and more direct, way by solving the linear system of equations (6) and neglecting the information $r = -x^T M x$.

When $m \geq 5$, the solution is unique (expect in degenerate cases). Then, elimination of r is possible by considering differences between squared measurements. Subtracting the i-th squared measurement equation from the m-th squared measurement equation (which is arbitrarily chosen without loss of generality) leads to

$$\begin{aligned} y_i^2 - y_m^2 - 2(y_i - y_m)\beta &= -(p_i^e - p_m^e)^T p_\Delta^e \\ &\quad + ||\breve{p}_i^e||_2^2 - ||\breve{p}_m^e||_2^2 \end{aligned} \quad (12)$$

for $i = 1, 2, ..., m-1$. Defining the vector $\delta \in \mathbf{R}^{m-1}$ by its elements

$$\delta_i \;:=\; y_i^2 - y_m^2 - ||\breve{p}_i^e||_2^2 + ||\breve{p}_m^e||_2^2$$

and re-arranging (12) leads to the Linear Time-Varying (LTV) measurement model

$$\delta \;=\; 2C_{\delta x} x \quad (13)$$

where $C_{\delta x} \in \mathbf{R}^{(m-1)\times 4}$ is defined by

$$C_{\delta x} \;=\; \begin{bmatrix} -(p_1^e - p_m^e)^T & y_1 - y_m \\ \vdots & \vdots \\ -(p_{m-1}^e - p_m^e)^T & y_{m-1} - y_m \end{bmatrix}$$

We then have the following result:

Lemma 2. *Suppose $m \geq 5$ and rank$(C_{\delta x}) = 4$. Then the unique solution to (6) is provided by the least squares estimate:*

$$x = \frac{1}{2} C_{\delta x}^{+} \delta \quad (14)$$

and $r = -x^T M x$ satisfies Lemma 1.

Proof. The result follows directly by observing that (13) can be obtained by eliminating r from (6) by Gaussian elimination of its last row. This elimination can be done since $m > \text{rank}(C_{\delta x}) = \dim(x)$, and hence we have at least one redundant measurement. □

Remark 1. *If solution ambiguity is not resolved by using either Lemma 2 or eq. (10), (e.g. when $m = 4$) one may sometimes use domain knowledge to exclude one of the position solutions. An example is provided by terrestrial satellite navigation where there is a large distance to the navigation satellites such that non-terrestrial solutions for the vehicle position can be ruled out. This can be done by setting $p_0^e = 0$ (reference point at the center of Earth using ECEF coordinate frame) and using an additional condition such as $||p_\Delta^e||_2 \approx r_e$, where r_e is the Earth's radius. Another example occurs in underwater navigation where all transponders are located on the seabed and the vehicle is at the surface, or least at some distance from the seabed, such that positions below the seabed can be ruled out. A third example is the use of bounds on β using knowledge of the measurement system's clock accuracy. This is motivated by the observation that the two alternative solutions typically have significantly different estimated values for β. A fourth example is to use additional sensors for depth or altitude.*

Next, we extend the quasi-linear measurement model to include range-rate measurements. The model can be derived from (2), using ideas similar to those described in [9]:

$$\nu_i(y_i - \beta) = (p_\Delta^e - \breve{p}_i^e)^T (v^e - v_i^e) \quad (15)$$

In this model we note the scalar bilinear term containing the two unknowns, $(p_\Delta^e)^T v^e$. This term can be eliminated by taking the difference between a pair of equations (15) since the term is common for all i. This leads to

$$\begin{aligned} \zeta_i \;&:=\; \nu_i y_i - \nu_m y_m - (\breve{p}_i^e)^T v_i^e + (\breve{p}_m^e)^T v_m^e \\ &=\; -(v_i^e - v_m^e)^T p_\Delta^e - (p_i^e - p_m^e)^T v^e + (\nu_i - \nu_m)\beta \end{aligned} \quad (16)$$

Stacking both pseudo-range and range-rate measurement equations in matrix format, we then obtain the model

$$\begin{pmatrix} \delta \\ \zeta \end{pmatrix} = \begin{bmatrix} 2C_{\delta x} & 0 \\ C_{\zeta x} & C_{\zeta v} \end{bmatrix} \begin{pmatrix} x \\ v^e \end{pmatrix} \quad (17)$$

where $\zeta \in \mathbf{R}^{m-1}$ and

$$C_{\zeta x} := \begin{bmatrix} -(v_1^e - v_m^e)^T & \nu_1 - \nu_m \\ \vdots & \vdots \\ -(v_{m-1}^e - v_m^e)^T & \nu_{m-1} - \nu_m \end{bmatrix},$$

$$C_{\zeta v} := \begin{bmatrix} -(p_1^e - p_m^e)^T \\ \vdots \\ -(p_{m-1}^e - p_m^e)^T \end{bmatrix}$$

The joint algebraic solution for position and velocity based on the quasi-linear measurement model is presented next.

Lemma 3. *Suppose $m \geq 5$ and define $C_R \in \mathbf{R}^{2(m-1)\times 7}$*

$$C_R = \begin{bmatrix} 2C_{\delta x} & 0 \\ C_{\zeta x} & C_{\zeta v} \end{bmatrix} \quad (18)$$

Assume rank$(C_{\delta x}) = 4$. Then the position and velocity are uniquely determined by

$$\begin{pmatrix} x \\ v^e \end{pmatrix} \;=\; C_R^{+} \begin{pmatrix} \delta \\ \zeta \end{pmatrix} \quad (19)$$

Proof. We note that $C_{\zeta v} \in \mathbf{R}^{(m-1)\times 3}$ satisfies rank$(C_{\zeta v}) = 3$ since the 3 columns of $C_{\zeta v}$ are also among the 4 linearly independent columns of $C_{\delta x}$. The result then follows directly since the above observation implies rank$(C_R) = 7$ and hence $C_R^T C_R$ is non-singular. □

A similar approach can be derived for the general case $m \geq 4$ by using the range measurement model (6).

Due to the elimination of prior knowledge contained in the nonlinear terms using redundant measurements, the linear least-squares estimates discussed above will generally be sub-optimal estimates (i.e. not achieve the Cramer-Rao lower bound) when there is measurement noise. In particular, the information about the nonlinear effects described by $r = -x^T M x$ are not utilized by the quasi-linear estimators.

IV. Second Stage Kalman Filter

Here we build on the above ideas and describe a KF based on a LTV measurement model. It will be used to estimate the position and velocity. We combine the quasi-linear time-varying measurement equations described in section III with a Linear Time-Invariant (LTI) model of the vehicle dynamics

$$\dot{p}^e_\Delta = v^e \tag{20}$$
$$\dot{v}^e = a^e + u_a \tag{21}$$

where u_a is assumed to be white noise. The above model also allows use of prior knowledge about the vehicle's acceleration a^e, which may be provided by an Inertial Measurement Unit (IMU), Attitude and Heading Reference System (AHRS), known control or environmental forces, other sensors or models, or set to zero when no such information is available.

Combining the vehicle dynamics (20)-(21) with (3), exact discretization gives a model

$$\chi_k = A\chi_{k-1} + Ba^e_{k-1} + Du_{k-1} \tag{22}$$

where the state is $\chi = (p^e_\Delta; \beta; v^e) = (x; v^e) \in \mathbf{R}^7$, the stochastic (unknown) input vector is $u = (u_a; u_\beta) \in \mathbf{R}^4$, $k \in \{1, 2, 3, ...\}$ is the discrete time index, and

$$A = \begin{bmatrix} I_3 & t_s I_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \quad B = \begin{bmatrix} (t_s^2/2)I_3 \\ 0 \\ t_s I_3 \end{bmatrix},$$
$$D = \begin{bmatrix} (t_s^2/2)I_3 & 0 \\ 0 & t_s \\ t_s I_3 & 0 \end{bmatrix}$$

where t_s is the sampling interval. A standard, time-varying, KF [22] is now used to estimate χ:

$$\begin{aligned} \bar{\chi}_k &= A\hat{\chi}_{k-1} + Ba^e_{k-1} \\ \bar{P}_k &= AP_{k-1}A^T + DQ_kD^T \\ K_k &= \bar{P}_k C_k^T (C_k \bar{P}_k C_k^T + R_k)^{-1} \\ \hat{\chi}_k &= \bar{\chi}_k + K_k(Y_k - \bar{Y}_k) \\ P_k &= (I - K_kC_k)\bar{P}_k \end{aligned}$$

where initial conditions $\bar{P}_0$ and $\bar{\chi}_0$ are given, and Q_k, R_k are noise covariance matrices of u_k and Y_k, respectively. These parameters depend on the choice of output $Y_k = C_k\chi_k$, its prediction $\bar{Y}_k$, and the measurement matrix C_k, which are treated differently in the general case (Section IV-A) and in the case with redundant measurements (Section IV-B) since the latter case allows some simplifications to be made. The following assumptions are made on the initialization and tuning of the KF:

Assumption 4. *$\bar{P}_0, Q_k, R_k > 0$ are arbitrary symmetric matrices.*

In Appendix VII, we compute the effects of measurement noise due to the nonlinear transforms in the quasi-linear measurement model. From these results, we can define a covariance matrix of the noisy measurement z' of the signal z:

$$\begin{aligned} R_{zz} &= \text{Cov}(z') \\ &= \text{diag}(2\sigma^4_{y1} + 4y_1^2\sigma^2_{y1}, ..., 2\sigma^4_{ym} + 4y_m^2\sigma^2_{ym}) \end{aligned}$$

and similarly for $R_{\zeta\zeta}$, $R_{\delta\delta}$, $R_{\delta\zeta} = R^T_{\zeta\delta}$, and $R_{z\zeta} = R^T_{\zeta z}$. We note that only R_{zz} is a diagonal matrix while the others are generally not, and that all terms generally depend on measurements y and ν. It also follows from the calculations in Appendix VII that zero-mean measurement noise on the pseudo-range y_i leads to a non-zero mean error on z_i. If all pseudo-range measurements y_i have the same white noise variance σ^2_y for all $i = 1, 2, ...m$, this bias would be embedded into the common error $\beta := c\Delta_c + \sigma^2_y$, and would cancel out when forming the differences δ_i.

The initialization $\bar{\chi}_0$ can, in principle, be arbitrary without loss of global convergence, although we recommend to use the algebraic solution in section III to achieve good estimation accuracy.

A. General case, $m \geq 4$

It is assumed here that the conditions of Lemma 1 are satisfied and that the quasi-linear time-varying measurement model (6) can be applied:

Assumption 5. *Transponders are located such that there exists a $\sigma^* > 0$ such that at all time k:*

$$C^T_{zw,k} C_{zw} \geq \sigma^* I_4$$

In this case, the variable $r = -(p^e_\Delta)^T p^e_\Delta + \beta^2$ appears in the measurement model (6) and should be estimated. In order to avoid introducing the nonlinearity into the error dynamics, or augmenting the state, we simply use the algebraic solution provided by Lemma 1 as a computed measurement of r. It is known from (9) that the choice of r is ambiguous and we must assume that this ambiguity can be resolved:

Assumption 6. *The ambiguity in r is resolved by using redundant transponders and (10), domain knowledge or additional measurements as discussed in Remark 1.*

We choose $Y_k = (z_k;\ \zeta_k)$, $\bar{Y}_k = C_k\bar{\chi}_k + E\bar{r}_k$ and

$$C_k = C_{G,k} = \begin{bmatrix} 2C_{zw} & 0 \\ C_{\zeta x} & C_{\zeta v} \end{bmatrix},$$

$$R_k = \begin{bmatrix} R_{zz} & R_{z\zeta} \\ R_{\zeta z} & R_{\zeta\zeta} \end{bmatrix},\quad Q_k = \text{Cov}(u_a; u_\beta)$$

where $E = (-\ell; 0)$ and $\bar{r}_k$ is the non-ambiguous algebraic estimate from (9) that in the absence of measurement noise is a perfect computed measurement satisfying $\bar{r}_k = r_k$. This leads to the filter structure in Figure 2. We then have the following key result regarding the 2nd stage estimates:

Proposition 1. *Let Assumptions 1 - 6 hold, then P_k is uniformly bounded, and the origin of the error dynamics of $\tilde{\chi}_k = \chi_k - \hat{\chi}_k$ is uniformly globally asymptotically stable (UGAS), except for the effects of the white noise.*

Proof. We have $Y_k - \bar{Y}_k = C_k\tilde{\chi}_k$. Uniform complete observability of $(A, R^{1/2}C_k)$ follows directly from the observation that the 3 columns of $C_{\zeta v}$ are linearly independent combinations of the $m \geq 4$ columns of C_{zw}. Hence, $\text{rank}(C_{\zeta v}) = 3$ and $\dim(Y_k) = \text{rank}(C_k) = 7$. Then the result follows from the standard properties of the Kalman-filter, [22], since $(A, DQ_k^{1/2})$ is controllable. □

B. Redundant information, $m \geq 5$

For this case when there are redundant measurements, i.e. when $m \geq 5$ and the conditions of Lemma 3 are satisfied, it is preferable to use the LTV measurement model (17) instead of (6), since there is no ambiguity issue in this case. We introduce:

Assumption 7. *Transponders are located such that there exists a $\sigma^* > 0$ such that, at all times k:*

$$C_{\delta x,k}^T C_{\delta x} \geq \sigma^* I_{m-1}$$

Then the variable r is eliminated and does not need to be estimated, cf. Figure 3. Hence, we can still use the KF, but now with $Y_k = (\delta_k;\ \zeta_k)$, $C_k = C_{R,k}$, $\bar{Y}_k = C_k\bar{\chi}_k$, and

$$R_k \quad = \quad \begin{bmatrix} R_{\delta\delta} & R_{\delta\zeta} \\ R_{\zeta\delta} & R_{\zeta\zeta} \end{bmatrix}$$

Under these conditions, we have:

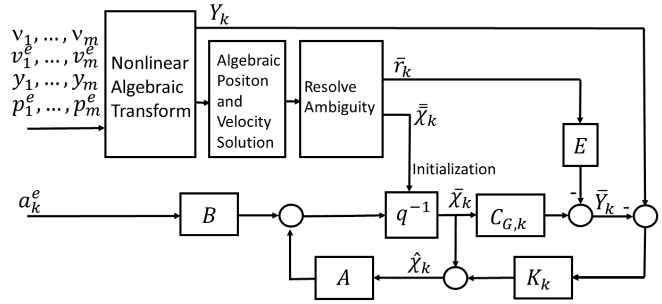


Fig. 2. Filter structure for the general case, when $m \geq 4$.

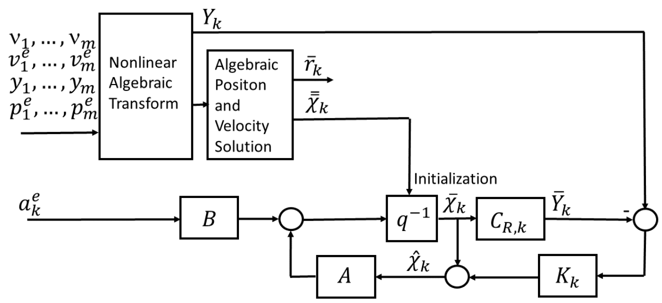


Fig. 3. Filter structure for the redundant case, when $m \geq 5$.

Proposition 2. *Let Assumptions 1 - 4 and 7 hold, then P_k is uniformly bounded, and the origin of the error dynamics of $\tilde{\chi}_k = \chi_k - \hat{\chi}_k$ is UGAS, except for the effects of the white noise.*

Proof. Uniform complete observability follows directly from the arguments of Lemma 3, and the result follows from the standard properties of the Kalman-filter, [22]. □

C. Additional comments

If the number of available range measurements, m, changes during operation, it is noted that the switching between the estimators is bumpless because the state space of the filter does not change, only its measurement structure.

Range-rate measurements are optional since observability of the vehicle dynamics (20)-(21) is straightforward to establish also when such measurements are not available.

It should be noted that Assumption 5 (for $m \geq 4$) and 7 (for $m \geq 5$) are sufficient, but not necessary, for uniform complete observability. Hence, if at some point in time one of these assumption does not hold, one should know that a less restrictive condition for uniform complete observability is given by the existence of an integer $N \geq 0$ and scalar $\varepsilon > 0$ such that

$W^o_{k,k+N} \geq \varepsilon I$ for all k, [22], where the observability Gramian is defined as

$$W^o_{k;k+N} = \sum_{i=0}^{N} (A^T)^{N-i} C_{k-i}^T C_{k-i} A^{N-i} \quad (23)$$

This condition may also be satisfied for $m < 4$, but will depend on the vehicle's trajectory.

Remark 2. *If observability is very weak (the Gramian being close to singular), or the above mentioned condition is violated for prolonged periods of time, the observability and performance might be improved by temporarily fixing the slowly time-varying β, or utilizing additional measurements to correct the estimates or enhance the prediction capabilities of the dynamic vehicle model.*

When defining the difference between measurements, cf. (12) and (16), it has be implicitly assumed that the different range and range-rate measurements are synchronized in time and simultaneously valid. Many range measurement systems, however, process the data asynchronously and may output one range at a time leading to latency and synchronization issues. In applications where the vehicle has high speed or accelerations, and the sampling frequency is relatively low, these latencies can lead to significant error in the data provided by the nonlinear transformation in the first stage. One might imagine that some compensation for these latencies can be achieved by integrating inertial measurements or range-rate measurements, or some other extrapolation method, in order to help synchronize the measurements in time. Moreover, the third stage linearized KF applies the individual raw measurements, and can process measurements sequentially as described in [5]. In this case, small latency errors are anticipated to lead to negligible performance loss since errors arising from lack of synchronization in the first stage will only influence the system matrices appearing in the model linearization in the second stage and the KF is relatively robust to errors in the system matrices.

V. Third Stage Kalman Filter

As discussed above, the transformation of the nonlinear model to a quasi-linear time-varying model implies some information loss since the prior knowledge provided by the nonlinear relationship (7) between the variables has been eliminated. This will generally lead to sub-optimal estimation in the presence of measurement noise. In order to recover better accuracy, while maintaining the UGAS of the estimates, a third estimation stage is proposed. In the third stage, the second-stage estimate $\hat{\chi}_k = (\hat{p}^e_{\Delta,k}; \hat{\beta}_k; \hat{v}^e_k)$ is used as the linearization point of a first order Taylor-series approximation for the original nonlinear measurement model (1) and (2) using $\mathcal{Y}_k = (y_k; \nu_k)$ as the measurement vector:

$$\mathcal{Y}_k \approx \hat{\mathcal{Y}}(\hat{\chi}_k) + \hat{\mathcal{C}}_k \left(\mathcal{X}_k - \hat{\chi}_k\right) \quad (24)$$

where $\hat{\mathcal{Y}}(\hat{\chi}_k)$ is an estimate of y_k based on $\hat{\chi}_k$, $\mathcal{X}_k$ is an instance of the state vector with the same structure as χ_k, and the linearized measurement matrix is

$$\hat{\mathcal{C}}_k = \begin{bmatrix} \frac{\hat{p}^e_{\Delta,k}+p^e_0-p^e_1}{||\hat{p}^e_{\Delta,k}+p^e_0-p^e_1||_2} & 1 & 0 \\ \vdots & \vdots & \vdots \\ \frac{\hat{p}^e_{\Delta,k}+p^e_0-p^e_m}{||\hat{p}^e_{\Delta,k}+p^e_0-p^e_m||_2} & 1 & 0 \\ \frac{\hat{v}^e_k-v^e_1}{||\hat{p}^e_{\Delta,k}+p^e_0-p^e_1||_2} & 0 & \frac{\hat{p}^e_{\Delta,k}+p^e_0-p^e_1}{||\hat{p}^e_{\Delta,k}+p^e_0-p^e_1||_2} \\ \vdots & \vdots & \vdots \\ \frac{\hat{v}^e_k-v^e_m}{||\hat{p}^e_{\Delta,k}+p^e_0-p^e_m||_2} & 0 & \frac{\hat{p}^e_{\Delta,k}+p^e_0-p^e_m}{||\hat{p}^e_{\Delta,k}+p^e_0-p^e_m||_2} \end{bmatrix}$$

This linearized measurement-matrix is then used in the third-stage linearized KF together with the dynamic vehicle model (22):

$$\begin{aligned} \bar{\mathcal{X}}_k &= A\hat{\mathcal{X}}_{k-1} + Ba^e_{k-1} \\ \bar{\mathcal{P}}_k &= A\mathcal{P}_{k-1}A^T + D\mathcal{Q}_k D^T \\ \mathcal{K}_k &= \bar{\mathcal{P}}_k \mathcal{C}_k^T (\mathcal{C}_k \bar{\mathcal{P}}_k \mathcal{C}_k^T + \mathcal{R}_k)^{-1} \\ \hat{\mathcal{X}}_k &= \bar{\mathcal{X}}_k + \mathcal{K}_k \left(\mathcal{Y}_k - \hat{\mathcal{Y}}(\hat{\chi}_k) + \mathcal{C}_k \left(\bar{\mathcal{X}}_k - \hat{\chi}_k\right)\right) \\ \mathcal{P}_k &= (I - \mathcal{K}_k \mathcal{C}_k)\bar{\mathcal{P}}_k \end{aligned}$$

where $\mathcal{R}_k$ is the covariance matrix of $\mathcal{Y}_k$ and $\mathcal{Q}_k = Q_k$. Note that this is not an EKF (but a linearized KF) since the approximate linearized LTV model (24) is not derived by linearization about the linearized KF's own estimates $\hat{\mathcal{X}}_k$, but rather about the 2nd stage estimates $\hat{\chi}_k$ that are used as time-varying exogenous inputs into the 3rd stage estimator, cf. Figure 1. We emphasize that the estimate from the 2nd stage KF is used only for linearization, and the 3st stage estimator does not inherit the state estimate or covariance matrix from the 2nd stage. Hence, the linearization cannot destabilize the estimator and therefore UGAS of the cascade of the three stages follows from standard results on cascades of UGAS systems (see [23] for a formal proof of a more general result based on the theory in [24]).

VI. Example

Consider the problem of navigation during autonomous landing of a small fixed-wing uninhabited aerial vehicle (UAV). As an alternative to the use of differential and carrier-phase GNSS (e.g. [25]) it is of interest to investigate the use of low-cost radio navigation systems enabled by wireless network technology, [1].

Such radio systems typically provide measurement range up to 300-1000 m with range errors of about 10-50 cm, [26], [27], which may be sufficient for this purpose. Various principles for range measurement and time synchronization can be used, with different error characteristics. In this study we assume the pseudo-range measurement model described in Section II. These sensors typically do not have Doppler shift measurements, so range rate measurements are not assumed in these simulations.

Since the radio beacons need to be deployed at locations near the ground, the terrain of typical UAV landing sites implies small vertical separation of the beacons even when placing them on masts, leading to a poorly conditioned estimation problem. We note that in a challenging operation such as search and rescue, or a field surveillance mission, a small vertical separation of the beacons typically results from the need for a relatively flat landing field, need for maximum visual and radio line of sight to the UAV from the ground control station which is usually close to the landing field, and limited time for optimal planning and placing the beacons. Moreover, since the UAV will land at ground level, its altitude will be similar to the vertical position of several of the beacons near the end of the final approach, i.e. just before touching the ground. This means that baselines will be crossed. The simulation example presented below as been deliberately chosen as a challenging (yet realistic) scenario in order to clearly illustrate the potential benefits of the approach. In fact, the simulations show that the global convergence property of the proposed method is essential for safe navigation without any further sensors in the selected scenarios, as well as similar scenarios that are tested but not reported.

We assume 6 radio beacons are distributed within an area with up to 1000 m horizontal separation, and with up to 10 m vertical separation between the beacons:

$$p_1^e = (600; 60; 5),\ p_2^e = (800; -100; 5)$$
$$p_3^e = (0; -100; 15),\ p_4^e = (0; 60; 15)$$
$$p_5^e = (0; -30; 5),\ p_6^e = (0; 30; 5)$$

The beacons provide signals that are used for pseudo-range measurements with standard deviation $\sigma_y = 0.15\ m$ and sampling interval of $0.2\ s$, corresponding to a realistic hardware setup, [26], [27].

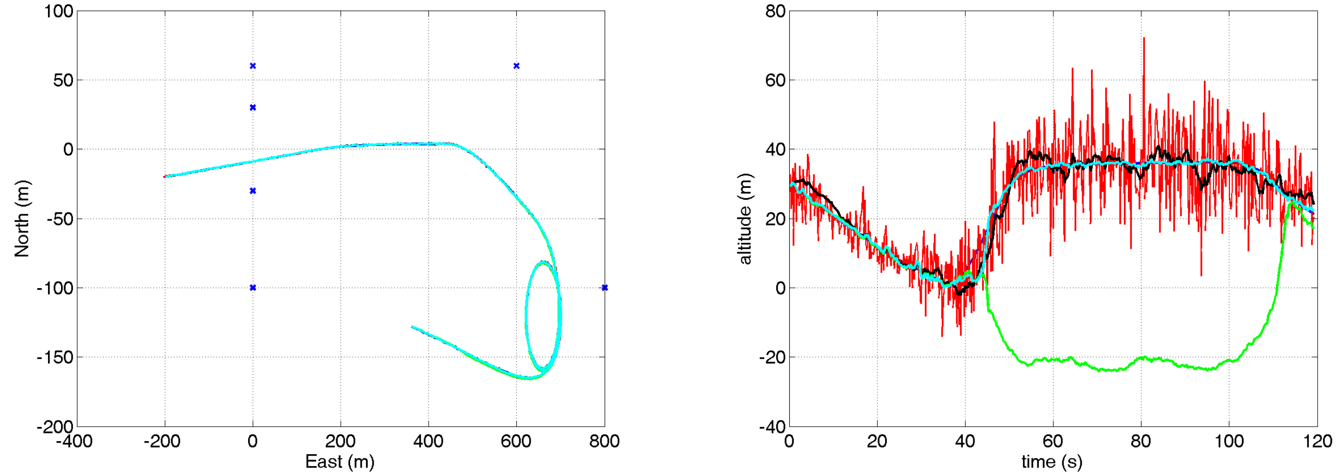
We simulate a small fixed-wing UAV trajectory in a final approach towards landing. Just before touching the ground, it is decided to abort the landing, so the vehicle climbs out and loiters at a holding position south of the landing target before going around to get in position for another approach. In a Monte-Carlo simulation with 100 scenarios, we simulate slight perturbations of the trajectory due to wind and different measurement noise realizations. Four different estimators are simulated and compared:

- Algebraic estimator (no filtering) based on the globally valid quasi-linear algebraic model (13) using weighted least squares (WLS) with $W = R_{\delta\delta}^{-1}$.
- 2nd stage global Kalman-filtering as described in section IV.
- 3rd stage Kalman-filtering, described in Section V.
- The EKF is considered to be a state-of-the-art method used as a benchmark [28], [6].

All KF's uses a stochastic model of the UAV dynamics. This means that no knowledge of acceleration a^e is assumed, i.e. the input is assumed to be white noise u_a. The covariance matrices of the white noise processes u_a and u_β are tuned empirically to achieve the best overall performance based on the analyzing several typical motions of the UAV: $Q_a = \text{diag}(50, 50, 2)$, $Q_\beta = 10^{-5}$, and the parameters of the measurement noise covariance matrices are according to Appendix VII.

During these simulations the EKF lost track of the correct global solution in 4 out of the 100 simulations. A representative example where this happens is shown in Figures 4 - 5. The horizontal position is accurately tracked by all estimators, and their differences in horizontal estimation accuracy can hardly been seen on this scale of presentation. However, the differences in performance are clearly apparent in the altitude estimate, see Figure 4b which shows that the 2nd stage KF provides less accurate estimates than the 3rd stage KF, and that the EKF diverges after $t = 42\ s$.

The reason for the failure of the EKF is the existence of multiple local minimums in the nonlinear least-squares criterion. When the UAV's altitude is close to the height of the beacons, at $h = 5\ m$ and $h = 15\ m$, it crosses their baselines. There, two local minimums are so close to each other that the EKF is not able to distinguish between them at some point near $t = 42\ s$, and chooses somewhat arbitrarily which minimum to track since there is conflicting information provided by the predictions of the dynamic model and the noisy measurements. It can be seen in Figure 6 that loss of track is, in this case, not linked to any worse geometric conditioning of the problem than at other times. It is remarked that the problem might have been reduced or avoided by having beacons at higher altitude or using additional sensors such as inertial sensors in order to improve the predictive capability of the model, or altimeter or GNSS to explicitly ensure the integrity of the system. It is notable that the KF based on the global algebraic quasi-



(a) Horizontal position (North-East)

(b) Vertical position (Altitude)

Fig. 4. True and estimated positions. The initial position is at -200 m East and -20 m North. The horizontal position of the six beacons are marked by blue points. Blue: Exact. Red: Algebraic WLS. Black: Second-stage KF. Cyan: Third-stage KF. Green: EKF. Note that the horizontal position estimates are very similar, and their differences cannot be clearly seen on the scale of this figure. The vertical position estimates are, however, clearly different.

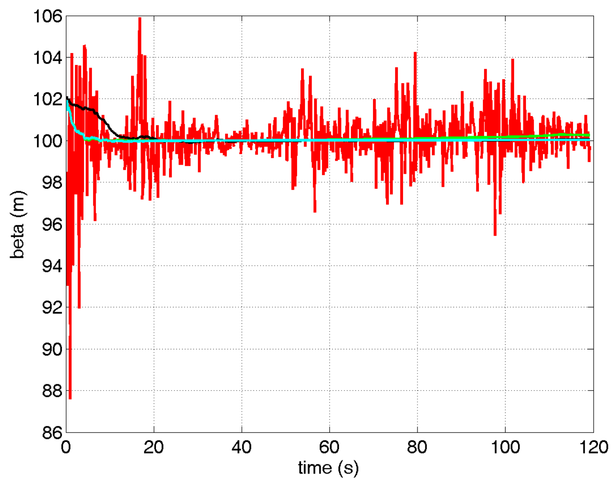


Fig. 5. Range bias β estimated with different observers. Blue: Exact ($\beta = 100$). Red: Algebraic WLS. Black: Second-stage KF. Cyan: Third-stage KF. Green: EKF. Note that the curves from the Third-stage KF and the EKF are close and difficult to distinguish on the figure.

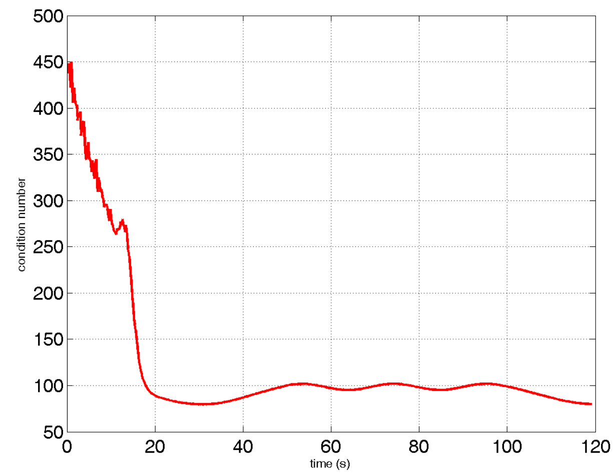


Fig. 6. Condition number of matrix $C_{\delta x}$, i.e. ratio between largest and smallest singular value.

TABLE I
MONTE-CARLO SIMULATIONS, AVERAGED OVER 96 CASES.

Estimator	Horizontal st.dev	Vertical st.dev
Algebraic WLS	1.4812	8.5375
2nd stage KF	0.52605	2.5753
3rd stage KF	0.30178	0.90865
EKF	0.30484	0.87357

linear model never loses track of the global solution, and consequently, the three-stage estimator never loses track of the global solution either.

More details can be seen in Figure 7, which shows the estimation errors. Moreover, average estimation errors for the 96 cases where the EKF does not diverge are summarized in Table I. It can be seen that the 3-stage estimator typically performs just as well as the EKF despite sub-optimal accuracy in the 1st and 2nd stages. Hence, for most of the time when the EKF does not experience divergence problems, the performances of the EKF and the three-stage estimator are very close.

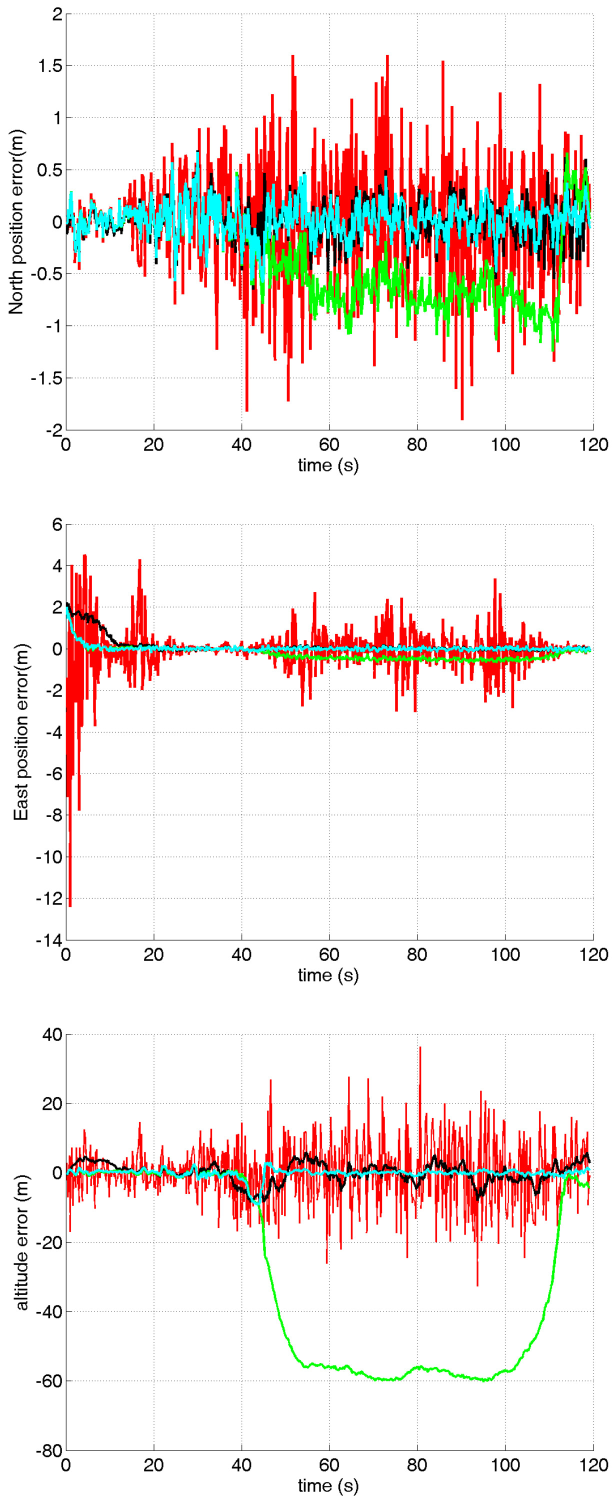


Fig. 7. North, East and vertical estimation errors. Blue: Exact. Red: Algebraic WLS. Black: Second-stage KF. Cyan: Third-stage KF. Green: EKF.

VII. Conclusions

The measurements of pseudo-range and range-rate are related to vehicle position and velocity through nonlinear measurement equations. It is well known that a global nonlinear algebraic transformation of the measurement equations and data leads to a globally valid quasi-linear time-varying measurement model. Employing this transformation as a 1st stage pre-processor of the measurements, the position and velocity can be estimated using standard Kalman-filtering using the quasi-linear time-varying measurement model in combination with a linear time-invariant model of the vehicle dynamics. Since the quasi-linear model is globally valid, this is a uniformly globally asymptotically stable filter. Whilst the algebraic transformation leads to a globally valid quasi-linear model in the noise-free case, the deliberate cancellation of nonlinearities entails some information loss which will generally lead to sub-optimal estimation accuracy in the presence of measurement noise. In order to recover better estimation performance, we propose another uniformly globally asymptotically stable estimator to be used in a 3rd stage using a linearized Kalman-filter employing a time-varying local linearization resulting from a Taylor-expansion of the original nonlinear model about the 2nd stage filtered estimates. This third-stage filter utilizes all the available information through the raw measurements and original measurement model, and hence offers the potential for high estimation accuracy.

An example has been presented, showing the effectiveness of the method at achieving both global convergence and good accuracy of the estimates when compared to an EKF. The method is particularly useful in applications where the vehicle crosses, or operates close to, baselines between transponders where ambiguities are not easy to resolve.

Acknowledgments

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Appendix A: Measurement noise properties

Let ε_i, e_i be independent normally distributed additive measurement noise on y_i, ν_i with zero mean and variance

σ_{yi}^2 and σ_{vi}^2. Consider the noisy computed outputs

$$\begin{aligned} z_i' &= (y_i+\varepsilon_i)^2 - ||\breve{p}_i^e||_2^2 \\ \delta_i' &= (y_i+\varepsilon_i)^2 - (y_m+\varepsilon_m)^2 - ||\breve{p}_i^e||_2^2 + ||\breve{p}_m^e||_2^2 \\ \zeta_i' &= (\nu_i+e_i)(y_i+\varepsilon_i) - (\nu_m+e_m)(y_m+\varepsilon_m) \\ &\quad -(\breve{p}_i^e)^T v_i^e + (\breve{p}_m^e)^T v_m^e \end{aligned}$$

Then we have for $i = 1, 2, ..., m$:

$$\begin{aligned} \mathbf{E}z_i' &= z_i + \sigma_{yi}^2 \\ \mathbf{E}(z_i' - \mathbf{E}z_i')^2 &= 2\sigma_{yi}^4 + 4y_i^2\sigma_{yi}^2 \end{aligned}$$

and for $i = 1, 2, ..., m-1$:

$$\begin{aligned} \mathbf{E}\delta_i' &= \delta_i + \sigma_{yi}^2 - \sigma_{ym}^2 \\ \mathbf{E}(\delta_i' - \mathbf{E}\delta_i')^2 &= 4y_i^2\sigma_{yi}^2 + 4v_m^2\sigma_{ym}^2 \\ \mathbf{E}\zeta_i' &= \zeta_i \\ \mathbf{E}(\zeta_i' - \mathbf{E}\zeta_i')^2 &= y_i^2\sigma_{vi}^2 + \nu_i^2\sigma_{yi}^2 + \sigma_{yi}^2\sigma_{vi}^2 \\ &\quad + y_m^2\sigma_{vm}^2 + \nu_m^2\sigma_{ym}^2 + \sigma_{ym}^2\sigma_{vm}^2 \end{aligned}$$

Cross-correlations $i \neq j$ are given by:

$$\begin{aligned} \mathbf{E}(z_i' - \mathbf{E}z_i')(z_j' - \mathbf{E}z_j') &= 0 \\ \mathbf{E}(\delta_i' - \mathbf{E}\delta_i')(\delta_j' - \mathbf{E}\delta_j') &= 4v_m^2\sigma_{ym}^2 \\ \mathbf{E}(\zeta_i' - \mathbf{E}\zeta_i')(\zeta_j' - \mathbf{E}\zeta_j') &= y_m^2\sigma_{vm}^2 + \nu_m^2\sigma_{ym}^2 + \sigma_{ym}^2\sigma_{vm}^2 \\ \mathbf{E}(\delta_i' - \mathbf{E}\delta_i')(\zeta_j' - \mathbf{E}\zeta_j') &= 2y_m\nu_m\sigma_y^2 + 2y_i\nu_j\mathbf{E}\varepsilon_i\varepsilon_j \\ \mathbf{E}(z_i' - \mathbf{E}z_i')(\zeta_j' - \mathbf{E}\zeta_j') &= 2y_i\nu_j\mathbf{E}\varepsilon_i\varepsilon_j + 2y_i\nu_m\mathbf{E}\varepsilon_i\varepsilon_m \end{aligned}$$

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BIOGRAPHY

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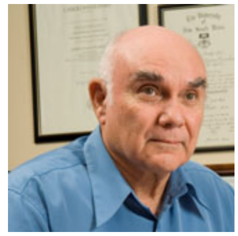

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